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BY

J. L. WALSH

PROFESSOR OF MATHEMATICS IN HARVARD UNIVE

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PREFACE

The general theory of the representation of functions by series of polynomials or more generally by series of other rational functions is far too vast to be included in a single treatise. Indeed, this theory can be considered to include the entire field of Fourier series as well as the detailed study of approximation of real functions by polynomials or by trigonometric polynomials. In the present work we restrict ourselves to the representation of functions in the complex domain, particularly analytic functions, by sequences of polynomials or of more general rational functions whose poles are preassigned, the sequences being defined either by interpolation or by extremal properties (i.e. best approximation). Taylor's series plays a central rôle in our entire study, for it has properties of both interpolation and best approximation, and serves as a guide throughout the whole treatise. Indeed, almost every result we give on the representation of functions is concerned with a generalization either of Taylor's series or of some property of Taylor's series—the title "Generalizations of Taylor's Series" would be appropriate.

In spite of this central character, we do not give an exposition of the modern detailed theory of Taylor's series (summability, gap theorems, Abelian and Tauberian theorems, overconvergence in the sense of Ostrowski, etc.); such accounts exist elsewhere in the literature [for instance, Dienes, 1931], and the theory of the other series which we study in detail has not yet reached the stage where these more delicate properties can be broadly treated. Other intentional omissions are necessary both to keep the size of the book within reasonable bounds and to preserve some unity of subject-matter and method. Such omissions are: factorial series and series of faculties; interpolation as related to the properties of entire functions; series of polynomials defined as solutions of differential or difference equations or defined by recursion formulas; continued fractions; expansions not found by interpolation or by an extremal property (Faber's polynomials are therefore omitted), except that the *possibility* of uniform approximation is conveniently studied by methods involving neither interpolation nor extremal properties. Every topic pertaining primarily to the real domain is automatically excluded. The theory of approximation of harmonic functions by harmonic polynomials or harmonic rational functions is analogous to the theory here set forth; for more specific information the reader who is interested in that theory may refer to a fairly recent report [Walsh, 1929a].

The history of our subject proper begins with Cauchy and the Cauchy-Taylor series. Later major contributions are due to Jacobi [1856], who introduced another important series of polynomials found by interpolation; to Runge [1885], who studied developments in series of polynomials or of rational functions valid in more or less arbitrary regions; to Hilbert [1897], who showed a connection between Green's function and approximation by polynomials; to Faber [1903],

who introduced a widely useful set of polynomials belonging to a region and who later [1920, 1922] studied in detail certain sequences of polynomials of best approximation; to S. Bernstein [1912], whose results on approximation in an interval of the axis of reals suggested results and methods appropriate to the complex domain; to Fejér [1918], who sharpened Hilbert's results and extended his methods; to Szegö [1921], whose memoir on polynomials orthogonal on a curve in the complex plane has become classical. The subsequent theory is especially the product of the last decade, much of it dependent on Montel's theory of normal families and on Carathéodory's theory of the conformal mapping of variable regions; important topics involved are: possibility of approximation by polynomials and by rational functions, further study of approximation by polynomials which have various extremal properties (best approximation), degree of convergence and overconvergence, interpolation and best approximation by rational functions with preassigned poles, extremal problems for analytic functions.

The theory presented in this book is by no means complete, but has nevertheless reached a certain degree of completeness. Topics have been selected which are as much as possible unified both as to method and content; for instance our study of interpolation and approximation by rational functions is in many respects a generalization of our study for polynomials. It is in accord with the traditions of this Colloquium Series that a writer should emphasize his own researches. In method we make large use of Cauchy's integral formula, corresponding formulas of interpolation, harmonic functions and conformal mapping, Schwarz's lemma and its generalizations, degree of convergence and its applications. The theory is fairly complete so far as concerns approximation to functions *analytic* on a given closed set, with results on *regions* of uniform convergence of the corresponding expansions. If the given functions are not analytic on the closed set considered, but for example are analytic in the interior of a region with merely certain continuity properties on the boundary, the questions of convergence on the boundary and degree of convergence are much more delicate, and have not yet been treated in any systematic manner.

As a matter of exposition, a few theorems from other parts of the theory of functions and which are frequently needed here are proved. But the main topic of the present book is interpolation and approximation, so theorems in other fields which occur in well known treatises are ordinarily not proved here. In some of the more special parts of the present work we employ without proof some theorems which appear only in the periodical literature. The treatment is therefore systematic, on the whole, and the book is intended both for the beginner and the specialist. The former can profitably omit the more special parts of the book on first reading; in particular he is advised to commence directly with Chapter III and to turn to Chapters I and II only when occasion requires.

References to the literature are inserted in the text. Figures in square brackets are dates, and indicate papers for which detailed references are given

in the Bibliography. Due to lack of space, the Bibliography is necessarily incomplete, even for the main topic of the book. The writer has recently written an essay *Approximation by polynomials in the complex domain* (Mémorial des Sciences Mathématiques). The general arrangement of material in the present book is similar to the arrangement in that essay, but the book undertakes to give a systematic exposition, not merely an outline. Many topics relating to polynomial approximation are mentioned in the essay without being mentioned in the present work, and on the other hand the essay does not consider interpolation and approximation by rational functions which are not polynomials. The book includes additional material developed in the two years since the essay was written. The reader who is interested primarily in interpolation and approximation by *polynomials* may find that essay useful for orientation.

Many results of this book have not been previously published, although some of them have been stated without proof or published in abstract in the essay just mentioned, in the Bulletin of the American Mathematical Society, in the Proceedings of the National Academy of Sciences, or in the Paris Comptes Rendus. Of these new results, far too numerous to be detailed here, we mention two. Chapter V includes by a single method all results known on the widely studied problem of regions of convergence of polynomials of best approximation to functions analytic on a given closed point set, except for best approximation in the sense of least squares on a line segment or circle with respect to certain pathological weight functions. The present results are sufficiently far-reaching to include some new results even for approximation in the sense of least squares (series of orthogonal polynomials) with a general weight function on a single interval of the axis of reals. Chapters VIII and IX contain unified results both old and new but more particularly unified methods of great generality on interpolation by rational functions, a topic on which the literature contains a large number of unrelated and more or less fortuitous discussions.

The writer has been most fortunate in receiving aid and encouragement in the writing of this book. Dr. L. Kalmár kindly furnished a translation into German of his thesis [1926], originally published in Hungarian. Dr. Y. C. Shen gave permission for the use of unpublished material from his Harvard thesis of 1935. The writer received a grant from the Milton Fund of Harvard University for help with the manuscript and proof, work that has been ably and admirably done by Miss Helen G. Russell. During the academic year 1934–35 the writer was in residence at Princeton as recipient of a grant from the Institute for Advanced Study. For all of this coöperation the writer expresses his deep gratitude, and also to the American Mathematical Society for accepting the book for publication in the Colloquium Series.

J. L. Walsh.

May, 1935

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CHAPTER I

POSSIBILITY OF APPROXIMATION; ANALYTIC FUNCTIONS

§1.1. Point sets: preliminary definitions

We shall be concerned in the present work entirely with the plane of the complex variable $z = x + iy$, or its stereographic projection onto the sphere. In the study of approximation by polynomials, we shall deal primarily with approximation in the *finite plane* (plane of finite points), that is, the plane without the adjunction of the point at infinity. In the study of approximation by more general rational functions, we shall ordinarily deal with the *extended plane*, that is, the plane with the adjunction of a single point at infinity. The extended plane is conveniently interpreted and studied by projection onto the sphere. If the type of plane is not explicitly mentioned, results are valid for either the finite plane or the extended plane.

Even though the extended plane is frequently used, we do not ordinarily allow infinity as a functional value or value of the dependent variable, except in connection with a geometric application involving the point at infinity. Under certain circumstances, which we shall mention explicitly, the wider convention is desirable, with the correspondingly wider definition of limit, continuity, and convergence.

A *point set* C is an arbitrary aggregate of points which *belong* to or are *contained* in C. Its *complement* consists of all points of the plane not belonging to C. The complement of C with respect to a set C_1 containing C consists of all points of C_1 not in C. A *neighborhood* of a finite point P is the interior of a circle whose center is P; a neighborhood of the point at infinity is the exterior of a circle in the extended plane. A *deleted neighborhood* of a point P is a neighborhood of P with the omission of P itself. A *limit point* of C is a point P (whether belonging to C or not) in every deleted neighborhood of which lie points of C. A point of C is *isolated* if it is not a limit point of C. A set C each of whose points is a limit point of C is said to be *dense in itself*. The set C' of limit points of an arbitrary set C is called the *derivative* or *derived set* of C. The derivative C'' of C' is called the *second derivative* of C, and similarly for higher derivatives. A set C one of whose derivatives $C', C'', C''', \cdots$ is empty is said to be *reducible*.

A *boundary point* of an arbitrary set C is a point P in every neighborhood of which lie points of C and points of the complement of C; the aggregate of all boundary points of C is the *boundary* of C. A point P is *exterior* to C if there exists some neighborhood of P containing no point of C. A point P is *interior* to C or is an *interior point* of C if there exists some neighborhood of P containing only points of C. The terms exterior and interior are also used in other senses (see below), particularly in connection with Jordan curves. A set C which

contains its limit points is *closed*. A set C whose elements are all interior points of C is *open*. A set C may be neither closed nor open. The derivative and the boundary of an arbitrary set are both closed. The complement of a closed set is an open set. The complement of an open set is a closed set.

A set is *limited* or *bounded* if it lies interior to some circle of the plane. A set which is not limited is *unlimited* or *unbounded*. The *distance between two sets* C_1 and C_2 is the greatest lower bound of all distances P_1P_2, where P_1 belongs to C_1 and P_2 belongs to C_2. The distance between two closed sets with no common element and at least one of which is limited, is positive. A set of points or other elements is *countable* or *denumerable* if the elements are finite in number or can be put into one-to-one correspondence with the set of positive integers.

A *Jordan arc* of the finite plane is a one-to-one continuous transform of a line segment, that is, a point set which can be represented by

$$(1) \qquad x = f_1(t), \qquad y = f_2(t), \qquad 0 \leqq t \leqq 1,$$

where $f_1(t)$ and $f_2(t)$ are continuous functions of t and where the system (1) has at most one solution t for given x and y. This Jordan arc is said to be *analytic* if $f_1(t)$ and $f_2(t)$ are analytic functions of t, for $0 \leqq t \leqq 1$ (see below), and if $|f_1'| + |f_2'| \neq 0$. A Jordan arc of the extended plane is a Jordan arc of the finite plane or any transform of such an arc by a transformation $w = 1/(z - \alpha)$; an analytic Jordan arc of the extended plane is similarly related to an analytic Jordan arc of the finite plane. If a Jordan arc AB meets a closed set C, there is a *first* point from A on the arc which belongs to C, and also a last point.

A point set C is *connected* if any two points of C can be joined by a Jordan arc consisting only of points of C. A *region* is an open connected set. Any two points of a region R can be joined by a broken line consisting of a finite number of segments all of whose points belong to R. A *closed region* or *domain* need not be a region, but is a region closed by the adjunction of its boundary points. If a region or other point set is denoted for instance by C, C_1, C', or R, we shall frequently without special mention of the fact denote by $\bar{C}, \bar{C}_1, \bar{C}'$, or $\bar{R}$ the corresponding region or point set closed by the adjunction of its boundary points.

A *Jordan curve* of the finite plane is a one-to-one continuous transform of a circumference, that is, a point set which can be represented by

$$(2) \qquad x = f_1(\theta), \qquad y = f_2(\theta),$$

where $f_1(\theta)$ and $f_2(\theta)$ are continuous functions of θ with period 2π, and where any two solutions θ of the system (2) for given x and y differ by an integral multiple of 2π. The Jordan curve is said to be *analytic* if $f_1(\theta)$ and $f_2(\theta)$ are analytic functions of θ, for $0 \leqq \theta \leqq 2\pi$, and if $|f_1'(\theta)| + |f_2'(\theta)| \neq 0$. We shall use the term *contour* in the sense of Jordan curve of the finite plane composed of a finite number of analytic Jordan arcs; a contour is rectifiable. A Jordan curve of the extended plane is a Jordan curve of the finite plane or any transform of such a curve by a transformation $w = 1/(z - \alpha)$, and is analytic if the original curve is analytic.

A Jordan curve C of the finite plane is known to separate the plane into precisely two regions, one limited (finite) and the other not limited (infinite), respectively the *interior* and *exterior* of C. We shall frequently have to deal with a point set Γ consisting of a finite number of Jordan curves of the finite plane, mutually exterior except that each of a finite number of points may belong to several of those curves. We shall use the terms *interior* and *exterior* of Γ to indicate respectively the set of all points each of which is interior to one of the Jordan curves composing Γ, and the set of all points each of which is exterior to all such curves.

A *Jordan region* of the finite plane is a limited (finite) region bounded by a Jordan curve. A Jordan region of the extended plane is a region of the extended plane bounded by a Jordan curve of the extended plane. A Jordan arc cannot separate the plane into two or more distinct regions.

If z is a complex number, the conjugate complex number is usually denoted by $\bar{z}$.

A function $f(z)$ is *analytic* or *holomorphic* at a finite point z_0 if it can be expressed in the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

throughout a neighborhood of z_0. A function $f(z)$ is analytic at infinity if it can be expressed in the form

$$f(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

throughout a neighborhood of the point at infinity. These are properties of *local* analyticity. A function is *analytic on a point set* if it is analytic at each point of the set. A function is *meromorphic* on a point set if it is analytic at each point of the set except possibly for poles.

A function is *harmonic* at a finite point if in a neighborhood of that point it is continuous together with its first and second partial derivatives and satisfies Laplace's equation. A function is harmonic at infinity if it is harmonic and uniformly limited at all points exterior to some circle; such a function approaches a definite finite limit at infinity. A function is *harmonic on a point set* if it is harmonic at each point of the set.

A function which can be expressed in the form $a_0z^n + a_1z^{n-1} + \cdots + a_n$ is called a *polynomial in z* and in particular a polynomial of degree n; we do not assume $a_0 \neq 0$. A function which can be expressed in the form

$$\frac{a_0z^n + a_1z^{n-1} + \cdots + a_n}{b_0z^n + b_1z^{n-1} + \cdots + b_n},$$

where the denominator does not vanish identically, is called a *rational function of z* and in particular a rational function of degree n.

A function is said to be *univalent* (*schlicht*) on a point set if it assumes no value more than once on that set.

The term *integrable* refers, unless otherwise specified, to integration in the sens of Lebesgue.

§1.2. Function-theoretic considerations

The reader is supposed to be familiar with the elements of the theory of func tions. Nevertheless, a few results of that theory are of such importance in th sequel that they deserve to be precisely formulated and treated here.

PRINCIPLE OF MAXIMUM. *If the function $f(z)$ is analytic in a region R of th extended plane, then the modulus $|f(z)|$ cannot have a proper maximum at a interior point of R, and can have an improper maximum interior to R only if $f(z$ is identically constant in R. The modulus $|f(z)|$ can have a proper minimum int rior to R only at a zero of $f(z)$, and can have an improper minimum interior to R onl at a zero of $f(z)$ or if $f(z)$ is identically constant in R.*

By the existence of a *proper maximum* of $|f(z)|$ at a point $z = z_0$ we mean tha the inequality

$$|f(z)| < |f(z_0)| \tag{3}$$

shall hold throughout some deleted neighborhood of the point $z = z_0$. By th existence of an *improper maximum* of $|f(z)|$ at a point $z = z_0$ we mean that th inequality

$$|f(z)| \leqq |f(z_0)| \tag{4}$$

shall hold throughout some neighborhood of the point $z = z_0$. Proper an improper minima are similarly defined.

The principle is valid for a region R of the extended plane. It is convenient the proof, however, and is no loss of generality, to assume that every point of is finite; if R is the extended plane, the function $f(z)$ must be identically constan in any other case a suitable transformation $w = 1/(z - \alpha)$ can be chosen necessary to transform every point of R into a finite point.

If the function $f(z)$ is analytic and not identically zero in the region R, at proper or improper maximum of $|f(z)|$ we have $|f(z)| > 0$. In the neighbo hood of such a point z_0 the function $\log |f(z)|$ is the real part of the analyt function $\log f(z)$ and hence is harmonic. By Gauss's mean value theorem f harmonic functions, the function $\log |f(z)|$ can have no proper maximum at and can have an improper maximum at z_0 only if $\log |f(z)|$ is identically co stant in that neighborhood and hence identically constant throughout R. Th function $\log x$ increases with x, so the inequalities

$$\log |f(z)| < \log |f(z_0)|, \qquad \log |f(z)| \leqq \log |f(z_0)|,$$

are respectively equivalent to (3) and (4). Hence $f(z)$ can have no proper ma mum interior to R, and can have an improper maximum interior to R only

$\log |f(z)|$ is identically constant in R. In the latter case the analytic function $\log f(z)$ of which $\log |f(z)|$ is the real part is also identically constant in R, so $f(z)$ is likewise identically constant in R.

We have therefore established the first part of the Principle of Maximum. The second part follows at once by study of the new function $1/f(z)$.

COROLLARY 1. *If $f(z)$ is analytic in a region R of the extended plane, continuous in the corresponding closed region $\bar{R}$, and if on the boundary of R we have*

$$|f(z)| \leqq M, \tag{5}$$

then the inequality (5) *is valid throughout $\bar{R}$.*

The function $|f(z)|$ is continuous in $\bar{R}$, hence possesses a maximum in $\bar{R}$. If that maximum is on the boundary of R, the corollary is established. If that maximum is interior to R, it must be either a proper or improper maximum of $|f(z)|$. Only the latter can actually occur, and if it does occur, the function $|f(z)|$ is identically constant in R and in $\bar{R}$; the proof is complete.

A somewhat more general result is

COROLLARY 2. *Let $f(z)$ be analytic in a region R (of the extended plane) whose boundary is denoted by B. Suppose that for every sequence of points $z_1, z_2, \cdots$ in R approaching a point of B we have for the superior limit*

$$\overline{\lim_{k \to \infty}} |f(z_k)| \leqq M .$$

Then inequality (5) *is valid throughout the interior of R.*

The set of values of $|f(z)|$ taken on in R has a finite or infinite least upper bound. There exists a sequence of values $|f(z_k')|$, with z_k' in R, approaching this least upper bound, and there exists a subsequence $|f(z_k)|$ such that the points z_k approach a limit z_0. If z_0 is interior to R, the point z_0 is an improper maximum of $|f(z)|$, and our conclusion now follows from the Principle of Maximum. If z_0 is on the boundary of R, our conclusion follows directly from our hypothesis.

Corollaries 1 and 2 have obvious applications to the study of inequalities of the form

$$|f(z)| \geqq m > 0 .$$

Generalizations of the previous results, proved by the same methods, are expressed in

COROLLARY 3. *The Principle of Maximum and Corollaries* 1 *and* 2 *remain valid if $f(z)$ is no longer analytic in R but is locally single-valued and analytic in R except for isolated branch points, and if $|f(z)|$ is single-valued in R.*

A frequently used application of the Principle of Maximum, here of Corollary 1, is that a series or sequence of functions each analytic in a region R of the

extended plane and continuous in the corresponding closed region $\bar{R}$ and whicl converges uniformly on the boundary of R converges also uniformly in the closec region $\bar{R}$. The limit of the series or sequence is analytic in R, continuous in $\bar{R}$

Corollary 4. *If the functions $f_1(z)$ and $f_2(z)$ are analytic in a region R of th extended plane, and if $|f_1(z)|$ and $|f_2(z)|$ are equal at every point of R, then we car write $f_1(z) \equiv \kappa f_2(z)$, where κ is a suitable constant of modulus unity. If at a singl point z_0 of R we have $f_1(z_0) = \lambda f_2(z_0) \neq 0$, then we have $\kappa = \lambda$.*

If $|f_2(z)|$ is identically zero in R, the conclusion is trivial. In any other case there exists some subregion of R in which $f_1(z)/f_2(z)$ is analytic and of modulu unity, hence (Principle of Maximum) is a constant κ of modulus unity. Thei $f_1(z)/f_2(z)$ is throughout R the constant κ.

Another result of the theory of functions of great use in the sequel is

Hurwitz's Theorem. *Let the functions $f_n(z)$ be analytic in a closed region $\bar{R}$ and converge uniformly in $\bar{R}$ to the function $f(z)$, necessarily analytic in R an continuous in $\bar{R}$. Suppose $f(z)$ is different from zero on the boundary of R. The for n sufficiently large, the number of zeros of $f_n(z)$ in R (or in $\bar{R}$) is precisely th number of zeros of $f(z)$ in R (or in $\bar{R}$), if the zeros are counted according to thei multiplicities.*

Thus the uniform limit $f(z)$ in a region R of non-vanishing analytic function $f_n(z)$ vanishes at all points of R or at no point of R.

This theorem is formulated for the region R, but applies at once also to th neighborhood of an arbitrary point interior to the given region R at which $f($ either vanishes or not. The proof of Hurwitz's theorem is not difficult and i left to the reader. A proof can conveniently be given (and was indeed given b Hurwitz) by the use of

Rouché's Theorem. *If the functions $f(z)$ and $g(z)$ are analytic interior to contour C, continuous in the corresponding closed region, and if the inequality*

$$|f(z)| > |g(z)|$$

is valid for z on C, then the functions $f(z)$ and $f(z) + g(z)$ have the same number zeros interior to C, the zeros being counted according to their multiplicities.

Rouché's theorem is readily proved by studying the variation of the arg ment of $[f(z) + g(z)]/f(z)$ as z traces C. The formal details may be supplie by the reader.

§1.3. An open set as the sum of regions

We need to establish a few preliminary results on point-set theory.

Theorem 1. *An arbitrary open set S is the sum of a finite or infinite number mutually exclusive regions. The boundary of each region is contained in the boun ary of S.*

All points of S which can be joined to a particular point P of S by Jordan arcs contained in S form together with P a set R which is a region. All points of such a Jordan arc clearly belong to R. The set R does not consist only of P, for all points of some neighborhood of P belong to R. Any two points A and B of this set R can be joined to P, hence to each other by Jordan arcs each of whose points belongs to R. Two of the given Jordan arcs may meet other than at P, but the portions from A and B respectively to the first point of intersection from A form together a Jordan arc in R joining A and B. Any point Q of R is an interior point of R, for an entire neighborhood of Q belongs to S, hence to R. Then R is an open connected set and hence a region.

The region R can be defined by choosing any one of its points as the particular point P. Hence any two such regions which have a point in common are identical.

A boundary point D of R must be a boundary point of S, for in every neighborhood of D lie points of R, hence points of S. If in a neighborhood of D lie no points not in S, then that whole neighborhood, including D itself, must belong to R. The proof is complete.

Any set of mutually exclusive regions of the plane is denumerable. Indeed, let us divide the plane by the lines $x = m/2^k$, $y = n/2^k$, where k, m, and n are integers, and denote by Σ_k the totality of squares of sides $1/2^k$ bounded by these lines. The set of squares Σ_k is denumerable and hence the totality of squares $\Sigma_1, \Sigma_2, \cdots$ is denumerable. Each of the given regions must contain in its interior at least one of these squares. No square is contained in two or more of the given regions. Hence the given set of regions is denumerable

As a matter of convention, when we consider the separation of the plane by a closed set C, we refer entirely to the separation of points not belonging to C. Two such points are said to be *separated by* C if there exists no Jordan arc joining those points which does not meet C.

COROLLARY. *Every closed set C, not the whole plane, separates the plane into a finite or denumerably infinite number of mutually exclusive regions. Each boundary point of such a region is a point of C.*

The proof is immediate by virtue of Theorem 1, for the complement of C is an open set S whose boundary is contained in C.

A closed set is called a *continuum* if it is the whole plane or if it separates the plane into regions all of which are simply connected. A *component* of an arbitrary closed set C is a maximal (i e., contained in no larger) subset which is a continuum. Each point of C lies in one and only one component of C.

It is frequently convenient to represent a region as a sum of simpler regions:

THEOREM 2. *Let R be an arbitrary region which does not contain the point at infinity. Then there exists a sequence of limited regions $R_1, R_2, \cdots$ with the following properties*: 1) *the domain $\bar{R}_n$ is interior to R*; 2) *the domain $\bar{R}_n$ is interior to*

R_{n+1}; 3) *every point of* R *lies in some* R_n; 4) *the region* R_n *is bounded by a finite number of non-intersecting contours*; 5) *the connectivity of* R_n *is not greater than that of* R.

Divide the entire plane into squares Σ_k by the lines $x = m/2^k$, $y = n/2^k$ as above. Let P be a fixed finite point of R whose coordinates x and y are both irrational; such a point exists. Denote by $\bar{S}_k$ the point set (which may be empty) consisting of $\bar{\Sigma}_k(P)$, namely the square Σ_k containing P plus its boundary, provided that closed square belongs to R, plus all other closed squares $\bar{\Sigma}_k$ belonging to R at a distance from P not greater than k which together with $\bar{\Sigma}_k(P)$ compose a closed region $\bar{S}_k$. The set $\bar{S}_k$ is uniquely determined, for if a square $\bar{\Sigma}_k$ is given, either there does or does not exist a number of other such squares of R distant from P not greater than k forming together with the given $\bar{\Sigma}_k$ a closed region containing $\bar{\Sigma}_k(P)$. The adjunction of new squares $\bar{\Sigma}_k$ cannot exclude any of the old ones forming with $\bar{\Sigma}_k(P)$ a closed region.

The region S_k is not necessarily bounded by a finite number of *non-intersecting* contours, but we now modify S_k and obtain a region S_k' as follows. If B is a boundary point of S_k, and if the four horizontal and vertical segments of length $1/2^k$ which terminate in B are all composed of points not belonging to S_k, there shall be deleted from S_k all points of the square (with horizontal and vertical sides) whose center is B and sides $1/2^{k+1}$. Such a point B must be at a distance from the boundary of R not greater than $2^{-k+1/2}$. The new set S_k' is a region and is bounded by a finite number of non-intersecting contours. Indeed, if we start from any point of the boundary of S_k' and continue to trace the boundary without reversing sense, there must be a first point at which we commence to retrace the boundary. The part of the boundary traced meanwhile is a Jordan curve. No other part of the boundary can intersect that Jordan curve.

The region S_k' increases or remains unchanged when k increases.

Every point Q of R lies in some S_k'. In fact, there exists a Jordan arc J consisting wholly of points of R whose end-points are P and Q and which lies in some circle whose center is P. Let the distance of J from the boundary of R be denoted by δ. When $1/2^{k-1} < \delta 2^{1/2}$ and when $k - 2$ is greater than the maximum distance from P to a point of J, with $k > 1$, then every point of J is the center of a square of side $1/2^{k-1}$ which belongs to R and interior to a square of side $1/2^k$ which belongs to S_k. There can be no first point on J from P which lies on the boundary of S_k', so every point of J (in particular Q) lies interior to S_k'.

Let us now choose R_1 as the first S_k' which contains P. Choose R_2 as the next succeeding S_k' which contains R_1 in its interior; such a new S_k' surely exists, by the reasoning used above in connection with the curve J; or see Theorem 4 below. We continue thus, and arrive at the desired sequence $R_1, R_2, \cdots$.

It remains to show that the connectivity of R_n is not greater than that of R. The connectivity of each S_k' is not greater than that of R, for each bounding contour of S_k' which bounds a finite region not belonging to S_k' must contain

in its interior points of the boundary of R. There is a point of the boundary of R exterior to each bounding contour of S_k', for the point at infinity is not a point of R. Thus the boundary of R has at least as many components as has the boundary of S_k'. The proof is complete.

If the connectivity of R is finite, then R_n has that same connectivity for n sufficiently large. A Jordan curve J_1 which passes through P and separates a component of the boundary of R from the other components lies in R_n for n sufficiently large.

COROLLARY. *Theorem 2 is true for the extended plane if R is not the entire plane and if we omit the requirements that R should not contain the point at infinity and that each R_n should be limited; if this is done and if R contains the point at infinity, then each R_n can be chosen to contain the point at infinity.*

The proof is immediate, for the situation can be treated from Theorem 2 by the use of a transformation $z - \alpha = 1/(w - \beta)$, transforming the point $w = \infty$ of the given region R of the Corollary to a point $z = x + iy = \alpha$, where x and y are irrational, and transforming a point $w = \beta$ of the boundary of the given region R to the point $z = \infty$.

THEOREM 3. *Let R be an arbitrary open set of the extended plane, not the entire plane. Then there exists a sequence of sets of regions $R_1, R_2, \cdots$ with the following properties: 1) the closed set $\bar{R}_n$ is interior to R; 2) the closed set $\bar{R}_n$ is interior to R_{n+1}; 3) every point of R lies in some R_n; 4) the point set R_n consists of a finite number of mutually exterior regions $R_n^{(k)}$ and is bounded by a finite number of non-intersecting contours; 5) the connectivity of each region $R_n^{(k)}$ of R_n is not greater than the connectivity of the maximal region belonging to R and containing $R_n^{(k)}$.*

Let R be composed (in the sense of Theorem 1) of the maximal regions $R^{(1)}$, $R^{(2)}, \cdots$, and let the region $R^{(k)}$ be related to the regions $R_1^{(k)}, R_2^{(k)}, \cdots$ as in the Corollary to Theorem 2. To prove Theorem 3 we need merely set symbolically

$$\begin{aligned}
R_1 &= R_1^{(1)}, \\
R_2 &= R_2^{(1)} + R_1^{(2)}, \\
R_3 &= R_3^{(1)} + R_2^{(2)} + R_1^{(3)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots, \\
R_n &= R_n^{(1)} + R_{n-1}^{(2)} + \cdots + R_1^{(n)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}$$

If the regions $R^{(k)}$ are finite (say K) in number, the symbols $R_n^{(k)}$, $k > K$, are to be omitted.

THEOREM 4. *Let R be an open set of the extended plane and let the open sets R_n (R_n interior to R_{n+1}) have also the properties expressed in* 1) *and* 3) *of Theorem* 3. *Let C be a closed set contained in R. Then there exists some R_n which contains C.*

Assume the theorem false; we shall reach a contradiction. There exists a point P_1 of C not in R_1, a point P_2 of C not in R_2, $\cdots$, a point P_k of C not in R_k, and so on. The points P_k have a limit point P (finite or infinite) which belongs to C and hence to R. The point P lies in some R_k, say R_N, and so also do a neighborhood of P and an infinity of the points P_k. But by hypothesis $P_k (k \geqq N)$ does not lie in R_k and hence cannot lie in R_N, a contradiction.

§1.4. Expansion of an analytic function

The fundamental theorems on representation of analytic functions by series of polynomials and other rational functions, from which can be derived the fundamental theorems on approximation, were established by Runge [1885].

THEOREM 5. *If the function $f(z)$ is analytic on and within the contour Γ, then there exists a sequence of rational functions whose poles lie on Γ which converges to $f(z)$ interior to Γ, uniformly on any closed point set interior to Γ.*

Let C be a closed point set interior to Γ; Cauchy's integral is

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)\, dt}{t - z}, \qquad z \text{ on } C; \tag{6}$$

here, and in the sequel unless otherwise specified, the integral is to be taken in the positive sense with respect to the region involved, that is to say, the sense which keeps the region on the left. The integral in (6) is the limit as n becomes infinite of the sequence

$$g_n(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{f(t_k)\,(t_{k+1} - t_k)}{t_k - z},$$

where $t_1, t_2, \cdots, t_n, t_{n+1} = t_1$ are successive points of subdivision of Γ, so chosen that $| t_{k+1} - t_k |$ approaches zero uniformly with $1/n$. For simplicity in notation, the dependence on n of the points t_k is not indicated.

The fact that the integral in (6) can be approximated by functions of form $g_n(z)$ suggests the use of the latter functions in the proof of the theorem. We can write

$$g_n(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{t_k}^{t_{k+1}} \frac{f(t_k)\, dt}{t_k - z},$$

$$f(z) - g_n(z) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{t_k}^{t_{k+1}} \left[\frac{f(t)}{t - z} - \frac{f(t_k)}{t_k - z} \right] dt.$$

For t on Γ and z on C, the function $f(t)/(t - z)$ of the two variables t and z is

continuous and uniformly continuous. If $\epsilon > 0$ is given and if n is so chosen that the maximum distance $| t_{k+1} - t_k |$ is sufficiently small, we have

$$\left| \frac{f(t)}{t - z} - \frac{f(t_k)}{t_k - z} \right| < \epsilon$$

for all z in C and for all t on the arc (t_k, t_{k+1}) of Γ. If l denotes the length of Γ, we now have*

$$| f(z) - g_n(z) | < \frac{l\epsilon}{2\pi}.$$

Consider the regions $R_1, R_2, \cdots$ of Theorem 2, where now R is the interior of Γ. There exist rational functions $G_j(z)$ whose poles lie on Γ such that we have

$$(7) \qquad \begin{aligned} | f(z) - G_1(z) | &< 1, & z &\text{ in } R_1, \\ | f(z) - G_2(z) | &< 1/2, & z &\text{ in } R_2, \\ &\cdots\cdots\cdots & &\cdots\cdots, \\ | f(z) - G_j(z) | &< 1/j, & z &\text{ in } R_j, \\ &\cdots\cdots\cdots & &\cdots\cdots. \end{aligned}$$

In the region R_j are valid the j-th inequality and also all the succeeding inequalities. Any closed point set interior to Γ lies in some R_j, so the sequence $G_j(z)$ converges uniformly to $f(z)$ on such a closed set. The sequence $G_j(z)$ is the sequence desired.

If $f(z)$ is given analytic in a closed limited region $\bar{R}$ bounded by several contours, the conclusion and proof are valid without change; the integral in (6) is now to be taken over the complete boundary of the region R.

Further, for z exterior to such a region R, Cauchy's integral (6) represents the function zero, because the integrand is an analytic function of t for t in $\bar{R}$. It is still true that the new function $g_n(z)$ defined as above differs but slightly from the value of the integral (6), for z on any closed point set exterior to R, and it is true that the rational function $G_j(z)$ can be subjected not merely to inequalities (7) but also to the inequalities

$$(8) \qquad | G_j(z) | < 1/j, \qquad z \text{ in } R'_j,$$

where the point sets R'_j are the sets of regions of Theorem 3, referring to the *exterior* R' of the present R. Inequalities (8) are valid even though the point set R'_j is unlimited, for if we demand that (8) should be valid in a limited annular region exterior to R but separating R from the point at infinity, then (8) is also valid (Principle of Maximum) in the points not separated from the

* An alternate proof here is to define a function on Γ equal to $f(t_k)/(t_k - z)$ on the arc of Γ bounded by t_k (inclusive) and t_{k+1} (exclusive). This function is piecewise constant on Γ and approaches $f(t)/(t - z)$ as n becomes infinite, uniformly for t on Γ and for z on C. The sequence found by term-by-term integration over Γ with respect to t converges uniformly for z on C.

point at infinity by that annular region; inequality (8) is valid at infinity when $G_j(z)$ is suitably defined there.

As a consequence of the remark just made, a function analytic in two or in any finite number k of mutually exclusive closed regions each bounded by a finite number of non-intersecting contours can be simultaneously represented in those open regions by a sequence of rational functions whose poles lie on the boundaries, simply by addition of the corresponding k sequences of rational functions $G_j(z)$ for the respective regions; the restriction (arising in the proof) that the given regions be limited is readily disposed of by a suitable transformation $w = 1/(z - \alpha)$.

THEOREM 6. *If the function $f(z)$ is analytic in k mutually exclusive regions $S_1, S_2, \cdots, S_k$ of the extended plane each bounded by a finite number of non-intersecting contours, then there exists a sequence of rational functions whose poles lie on the boundaries of the S_j, which converges to $f(z)$ interior to the S_j, uniformly on any closed set interior to the S_j.*

We add the remark (probably familiar to the reader) which is general and irrespective of the present situation that *possibility of uniform approximation and possibility of uniform expansion are equivalent*; more explicitly, if a function $f(z)$ can be uniformly approximated as closely as desired on a point set C (more briefly, can be approximated on C) by a rational function of z or a polynomial in z, then there exists a sequence of rational functions of z or of polynomials in z converging uniformly to $f(z)$ on C; conversely, if there exists a sequence of rational functions of z or of polynomials in z converging uniformly to $f(z)$ on C, then $f(z)$ can be approximated on C by a rational function of z or by a polynomial in z. Proof of the first part of this remark is by use of inequalities such as (7) all valid for z on C; the sequence $G_j(z)$ converges uniformly to $f(z)$ on C. Proof of the second part is immediate by the definition of uniform convergence. For some purposes we shall find it convenient to use the concept of approximation, for other purposes the concept of uniform expansion.

§1.5. A theorem on analytic extension

The theorems just proved have two disadvantages: 1) the convergence to $f(z)$ of the sequence of rational functions is proved only *interior* to the given regions; 2) the poles of the rational functions lie automatically on the boundaries of the regions and cannot by the method just used be chosen to lie elsewhere.

The first of these objections is readily overcome:

THEOREM 7. *Let the function $f(z)$ be analytic on the closed set C (not the entire plane) of the extended plane. Then there exists a point set S consisting of a finite number of regions each bounded by a finite number of non-intersecting contours such that S contains C in its interior and such that a function $F(z)$ analytic on the closed set $\bar{S}$ coincides with $f(z)$ on C. In particular, if C consists of a finite number of mutually exterior closed regions and if C is bounded by a finite number of non-*

intersecting Jordan curves, then $\bar{S}$ can be chosen to consist of the same number of mutually exterior closed regions bounded respectively by the same numbers of contours.

Otherwise expressed, this theorem is concerned with the single-valued analytic extension of $f(z)$.

We take C as limited, which involves no loss of generality. If z is a point of C, we denote by $\rho(z)$ (finite or infinite) the radius of the largest circle within which the Taylor development of $f(z)$ at z converges and represents $f(z)$ on C. Of course $\rho(z)$ may be less than the radius of convergence of the Taylor development of $f(z)$ at z, for C may fall into several components and the function $f(z)$ defined on one component may bear no relation to the analytic extension of the function $f(z)$ defined on another component. If z' is an isolated point of C, we consider the Taylor development of the function $f(z)$ at z' to be identically $f(z')$, so that $\rho(z')$ is the distance from z' to the subset of C where $f(z)$ does not take the value $f(z')$. We shall prove that *the numbers $\rho(z)$ for all z in C have a positive lower bound δ.* We assume the contrary and shall reach a contradiction. There exists then a point z_k of C such that $\rho(z_k) < 1/k$. The points z_k have at least one limit point z_0, which belongs to C, and we have $\rho(z_0) > 0$. This contradicts for an infinity of numbers k the assumption $\rho(z_k) < 1/k$.

Denote by δ_1 the smaller of the two quantities, the number δ whose existence has just been proved and the number η, where η is the shortest distance from one component of C to another if C consists of a finite number of mutually exterior closed regions, and where η is otherwise infinite.

The set Σ composed of all points of the plane whose distances from C are not less than $\delta_1/3$ is closed. Apply Theorems 3 and 4 to the complement C_1 of C. Let S_1 be a set of regions in C_1 bounded by a finite number of non-intersecting contours containing Σ in its interior; and also, provided C is bounded by a finite number of non-intersecting Jordan curves, such that each region composing S_1 has the same connectivity as the maximal region of C_1 containing it, with one region of S_1 in each maximal region of C_1. Let S be the complement of S_1. Then $\bar{S}$ satisfies the topological requirements. The function $F(z)$ shall be equal to $f(z)$ on C, and otherwise at a point z of $\bar{S}$ shall be defined by the Taylor development of $f(z)$ at a point of C distant not more than $\delta_1/3$ from z (at least one such point of C exists). Then $F(z)$ is analytic at each point of $\bar{S}$ and is also single-valued, for if there are two points z_1 and z_2 of C distant not more than $\delta_1/3$ from z, they yield the same definition of $F(z)$ at z; a circle whose center is z_1 and radius δ_1 contains both z and z_2 in its interior, and $f(z)$ is analytic in such a circle. The proof is complete.

We emphasize the fact that the analytic extension of $f(z)$ is not uniquely defined in the neighborhood of an isolated point z' of C. Infinitely many functions are analytic in the neighborhood of z' and coincide with $f(z)$ on C. This remark is of significance in connection with Theorem 7 and also in connection with Theorem 15 below.

Theorem 7 has obvious application to approximation in connection with

Theorem 6. We postpone the statement of the new result until we have discussed in more detail the choice of poles of the approximating rational functions.

§1.6. Approximation; choice of poles

We turn now to the question of the location of the poles of the approximating rational functions, and we shall find it convenient to develop our results in the form of several lemmas.

LEMMA I. *If the closed point set C lies interior to a circle γ, if the function $\phi(z)$ is analytic on and within γ, and if $\eta > 0$ is arbitrary, then there exists a polynomial $p(z)$ in z such that we have*

$$| \phi(z) - p(z) | < \eta, \qquad z \text{ on } C.$$

The proof is immediate, for $p(z)$ need merely be chosen as the sum of a sufficiently large number of terms in the Taylor development of $\phi(z)$ about the center of γ.

LEMMA II. *If the closed point set C lies exterior to a circle γ, if the function $\phi_j(z)$ has its only singularity in the extended plane at a point z_j interior to γ, if z_{j+1} denotes another point interior to γ, and if $\eta > 0$ is arbitrary, then there exists a polynomial $p_j[1/(z - z_{j+1})]$ in the variable $1/(z - z_{j+1})$ such that we have*

$$| \phi_j(z) - p_j[1/(z - z_{j+1})] | < \eta, \qquad z \text{ on } C.$$

The proof follows directly from Lemma I by making the substitution $w = 1/(z - z_{j+1})$.

LEMMA III. *If the finite or infinite points $z = T$ and $z = Z$ lie in a region R, if the function $\phi(z)$ has as its only singularity in the extended plane the point T, and if $\epsilon > 0$ is arbitrary, then there exists a polynomial $P(z)$ in the variable $1/(z - Z)$ if Z is finite and in the variable z if Z is infinite such that we have*

$$| \phi(z) - P(z) | < \epsilon, \qquad z \text{ in the complement of } R.$$

We give the proof for the case that T and Z are both finite points; the more general case easily reduces to this. There exists a finite broken line α of a finite number of segments which lies interior to R and joins T and Z. Let d be less than the distance from α to the boundary of R, and let points $z_0 = T, z_1, \cdots, z_{m-1}$, $z_m = Z$ be chosen on α so that $| z_k - z_{k+1} | < d$. Let C_k be the circle whose center is z_k and radius d. The points $z_0 = T$ and z_1 lie in C_1; z_1 and z_2 in C_2; z_2 and z_3 in C_3; $\cdots$; z_{m-1} and $z_m = Z$ in C_m.

By Lemma II, there exist polynomials $p_k(z)$ in the respective variables $1/(z - z_k)$ such that we have

$$
\begin{aligned}
&| \phi(z) \quad - p_1(z) | < \epsilon/m, && z \\
&| p_1(z) \quad - p_2(z) | < \epsilon/m, && z \text{ in the } \ldots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&| p_{m-1}(z) - p_m(z) | < \epsilon/m, && z \text{ in the complement of } R.
\end{aligned}
$$

The polynomial $p_m(z)$ is then clearly the polynomial in $1/(z - Z)$ desired.

We are now in a position to state a number of important results:

THEOREM 8. *Let the function $f(z)$ be analytic on the closed set C of the extended plane. If $\epsilon > 0$ is arbitrary, then there exists a rational function $r(z)$ of z such that we have*

$$|f(z) - r(z)| < \epsilon, \qquad z \text{ on } C.$$

If points z_k (not necessarily denumerable) are chosen, at least one in each of the regions into which C separates the plane, then the poles of $r(z)$ can be chosen to lie in the points z_k. In particular, if C is a finite Jordan region or several mutually exterior finite Jordan regions, or more generally a closed limited point set whose complement is connected, then $r(z)$ can be chosen a polynomial in z.

The first part of this theorem follows at once by Theorem 7 and Theorem 6. The latter part is not difficult to prove Let C be finite for definiteness, together with the points z_k. Theorem 7 and Theorem 6 yield a rational function $g(z)$ such that we have

$$|f(z) - g(z)| < \epsilon/2, \qquad z \text{ on } C, \tag{9}$$

and the poles $\zeta_1, \zeta_2, \cdots, \zeta_N$ (which we assume for definiteness finite) of $g(z)$ lie clearly exterior to C:

$$g(z) = A_0 + \sum_{k=1}^{N} \frac{A_k}{z - \zeta_k}. \tag{10}$$

Corresponding to every point ζ_k there is at least one point z_j not separated from it by points of C Then by Lemma III there exists a polynomial $p_k(z)$ in $1/(z - z_j)$ such that we have

$$\left| \frac{A_k}{z - \zeta_k} - p_k(z) \right| < \frac{\epsilon}{2N}, \qquad z \text{ on } C. \tag{11}$$

Our theorem is now a consequence of (9), (10), and (11):

$$\left| f(z) - \left[A_0 + \sum_{k=1}^{N} p_k(z) \right] \right| < \epsilon, \qquad z \text{ on } C.$$

The general question of possibility of approximation is to be discussed in more detail later (especially §1.10), but it is worth while to notice now that, although a function $f(z)$ analytic in a *closed Jordan region* of the finite plane can be uniformly approximated in that closed region by a polynomial in z, *it is not true that*

the corresponding result holds for an arbitrary closed limited simply connected region. We prove this fact by the following illustration. The function $f(z)$ is chosen as $1/z$, the region C is chosen as a strip closed at one end, exterior to the unit circle γ: $|z| = 1$, winding infinitely many times about γ, and approaching γ. The function $f(z)$ is analytic in the closed region $\bar{C}$ and the circle γ belongs to that closed region. If the function $f(z)$ can be uniformly approximated by a polynomial in the closed region $\bar{C}$, there exists a sequence $p_n(z)$ of polynomials which converges uniformly to $f(z)$ in $\bar{C}$, hence in particular on γ. By the uniformity of this convergence on γ we have

$$\int_\gamma p_n(z)\,dz = \int_\gamma \frac{dz}{z}, \qquad 0 = 2\pi i,$$

which is absurd.

THEOREM 9. *Let the function $f(z)$ be analytic at every point of the open set R of the extended plane. Then there exists a sequence of rational functions converging to $f(z)$ at every point of R, uniformly on every closed set interior to R. If points z_j (not necessarily denumerable) are chosen at least one on or separated from R by each component of the boundary of R, then these rational functions can be chosen with their poles in the points z_j. In particular, if R is a simply connected region not containing in its interior the point at infinity, then these rational functions can be chosen as polynomials in z.*

The case that R is the entire extended plane is trivial and henceforth excluded.

Let $R_1, R_2, \cdots$ be the sets of regions of Theorem 3. At least one point z_j lies interior to each of the regions into which $\bar{R}_k$ separates the plane. Thus (Theorem 8) there exists a rational function $r_k(z)$ of the kind desired such that we have

$$\begin{aligned}
|f(z) - r_1(z)| &< 1, & z \text{ in } \bar{R}_1,\\
|f(z) - r_2(z)| &< 1/2, & z \text{ in } \bar{R}_2,\\
&\cdots\cdots\cdots\cdots, \\
|f(z) - r_k(z)| &< 1/k, & z \text{ in } \bar{R}_k,\\
&\cdots\cdots\cdots\cdots.
\end{aligned}$$

The sequence $r_k(z)$ satisfies the requirements of the theorem.

If desired, the points z_k can be made to depend on the index of the approximating rational function $r_n(z)$. It is sufficient if the points are prescribed in the form

$$\begin{gathered}
z_{11}, z_{12}, \cdots, \\
z_{21}, z_{22}, \cdots, \\
\cdots\cdots\cdots,
\end{gathered}$$

where all the poles of $r_n(z)$ must lie in the points z_{nk}, and where at least one point of the set $z_{n1}, z_{n2}, \cdots$ is on or separated from R by each component of the bound-

ary of R; it is sufficient if a point z_{nk} lies in each of the regions into which R_n separates the plane. It is also sufficient if the points are given as

$$\begin{array}{l} z_{11}, \\ z_{21}, z_{22}, \\ z_{31}, z_{32}, z_{33}, \\ \cdots\cdots\cdots, \end{array}$$

where again all the poles of $r_n(z)$ must lie in the points z_{nk}, provided the boundary of R has only a denumerable number of components $C_1, C_2, \cdots$ and the point z_{nk} is on or separated from R by C_k, or provided for n sufficiently large at least one point z_{nk} lies in each of the regions into which R_n separates the plane.

A similar remark clearly applies to Theorem 8.

§1.7. Components of an analytic function

Some of our previous results can be sharpened. Let the function $f(z)$ be analytic in a closed finite region $\bar{R}$ bounded by a finite number of mutually non-intersecting contours $C_1, C_2, \cdots, C_\lambda$. Then $f(z)$ is also analytic in a closed finite region $\bar{R}'$ bounded by a finite number of mutually non-intersecting contours $\Gamma_1, \Gamma_2, \cdots, \Gamma_\lambda$, where R' contains $\bar{R}$ in its interior, and where C_j separates Γ_j from $\bar{R}$. The function $f(z)$ can be expressed in $\bar{R}$ by Cauchy's integral:

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1+\Gamma_2+\cdots+\Gamma_\lambda} \frac{f(t)\,dt}{t-z}, \qquad z \text{ in } \bar{R}, \tag{12}$$

where the integral is taken in the positive sense with respect to R'. Each of the functions (*components* of $f(z)$)

$$f_j(z) \equiv \frac{1}{2\pi i}\int_{\Gamma_j} \frac{f(t)\,dt}{t-z} \tag{13}$$

is single-valued and analytic not merely in R but in the entire closed Jordan region $\bar{R}_j$ of the extended plane bounded by C_j and containing R; the proof may be given directly from (13), by showing that $f(z)$ as thus defined has a continuous derivative at every finite point not on Γ_j and is also analytic at infinity if defined to have the value zero there.*

If the point z_j is separated by C_j from R, then $f_j(z)$ can be uniformly approximated in $\bar{R}_j$ by a polynomial in $1/(z - z_j)$ if z_j is finite and by a polynomial in z if z_j is infinite.

We have established the following theorem for the case that R is finite; the contrary case is readily reduced to this.

* The components $f_j(z)$ are uniquely determined from $f(z)$ and R by the fact that they are analytic in the $\bar{R}_j$ respectively and vanish at infinity if $\bar{R}_j$ is infinite, and that their sum is $f(z)$ in R. Under a linear transformation of $f(z)$ and R, each component is altered at most by an additive constant.

THEOREM 10. *If the function $f(z)$ is analytic in the closed region $\bar{R}$ of the extended plane bounded by non-intersecting contours $C_1, C_2, \cdots, C_k$, then $f(z)$ can be expressed in $\bar{R}$ as the sum*

$$f(z) \equiv f_1(z) + f_2(z) + \cdots + f_k(z), \tag{14}$$

where $f_j(z)$ is analytic in the closed Jordan region $\bar{R}_j$ bounded by C_j containing R in its interior. If the point z_j is separated from R by C_j, then $f_j(z)$ can be uniformly approximated in $\bar{R}_j$ by a polynomial in $1/(z - z_j)$ if z_j is finite and by a polynomial in z if z_j is infinite.

Theorem 10 is somewhat more specific than Theorem 8 as applied to a region. It is no effective restriction on the region R in Theorem 10 ($f(z)$ being assumed analytic in $\bar{R}$) that R is assumed bounded by a finite number of mutually non-intersecting contours, by virtue of Theorem 7.

A corresponding theorem can be proved in the same manner for expansion of a function analytic in a region:

THEOREM 11. *Let the function $f(z)$ be analytic in the region R of the extended plane bounded by the mutually exclusive closed point sets $C_1, C_2, \cdots, C_k$, and let $R_1, R_2, \cdots, R_k$ be the regions of the extended plane bounded by $C_1, C_2, \cdots, C_k$. respectively which contain R. Then $f(z)$ can be expressed in R as the sum* (14), *where $f_j(z)$ is analytic in R_j. In the region R_j the function $f_j(z)$ can be expressed as a series of rational functions of z converging uniformly on any closed point set interior to R_j; the poles $z_n^{(j)}(n = 1, 2, \cdots)$ of these rational functions can be chosen as prescribed points on C_j or separated from R_j by C_j, provided at least one point $z_n^{(j)}$ is on or separated from R_j by each component of C_j.*

If the region R does not contain the point at infinity in its interior, the function $f_j(z)$ can be defined at an arbitrary point of R_j by the integral (13), where now Γ_j is a contour which depends on z and is so chosen that it lies in R and separates z from C_j but not from any point of $C_1, C_2, \cdots, C_{j-1}, C_{j+1}, \cdots, C_k$. The sense of integration on Γ_j is to be chosen so that (14) is valid for z in R.

The point sets C_j need not be *components* of the boundary of R; in fact, the boundary of R may have a non-denumerable infinity of components.

From the continuity properties of the $f_j(z)$ we can read off the

COROLLARY. *If $f(z)$ is continuous on C_j, so are $f_1(z), f_2(z), \cdots, f_{j-1}(z), f_{j+1}(z), \cdots, f_k(z)$, and hence so also is $f_j(z)$.*

It is worth remarking that if the function $f(z)$ is expressed in the region R of Theorem 11 (R supposed finite) as the limit of a sequence $r_n(z)$ of rational functions whose poles lie exterior to R, then in any closed limited subregion $\bar{R}'$ any component $f_j(z)$ of $f(z)$ (i.e., in the sense of Theorem 10) is the uniform limit of the corresponding components $r_n^{(j)}(z)$ of the $r_n(z)$. The proof is immediate, by Cauchy's integral

$$f_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j}\frac{f(t)\,dt}{t - z}, \qquad r_n^{(j)}(z) = \frac{1}{2\pi i}\int_{\Gamma_j}\frac{r_n(t)\,dt}{t - z},$$

where Γ_j is a contour or suitable set of non-intersecting contours in R separating $\bar{R}'$ from C_j but not from C_m with $m \neq j$, and where z is in $\bar{R}'$, for $r_n(t)$ approaches $f(t)$ uniformly on Γ_j. Indeed, the function $r_n^{(j)}(z)$ approaches $f_j(z)$ not merely in $\bar{R}'$ but also in every point of R_j.

The results of §§1.3–1.7 are in the main due to Runge [1885], although he emphasizes the *expansion* of functions rather than approximation. So far as the writer is aware, Theorem 8 was first formulated by Walsh [1928a].

§1.8. Methods of Appell and of Wolff

Two other methods of approximation are of sufficient interest to deserve treatment. Like the method of Runge, both methods take Cauchy's integral as point of departure. We prove first the following theorem [Appell, 1882]:

THEOREM 12. *Let $C_1, C_2, \cdots, C_k$ be circular regions (a circular region is the interior or exterior of a circle, or a half-plane) bounded by the respective circles $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$. Let regions $R_1, R_2, \cdots, R_\nu$ bounded by the Γ_j be common to all the regions C_j, and let the function $f(z)$ be single-valued and analytic in each of the closed regions $\bar{R}_j$. Then the function $f(z)$ can be expanded in all the R_j in a series of rational functions, converging uniformly on any closed set interior to the R_j.*

The function $f(z)$ need not be a monogenic analytic function, but is merely to be single-valued and analytic as indicated. A given region R_j need not abut on *all* the C_m.

In the proof we suppose, as we may do with no loss of generality, that the regions R_j are all finite (it is sufficient if a single C_m is finite), and that the circles Γ_j are proper circles and not straight lines. The function $f(z)$ is represented interior to R (sum of the R_j) by Cauchy's integral for $f(z)$ taken over the boundary B of R:

$$f(z) = \frac{1}{2\pi i}\int_B \frac{f(t)\,dt}{t - z}, \quad z \text{ interior to } R; \tag{15}$$

this formula is of course valid even if ν is greater than unity. To be sure, the function $f(z)$ has not been supposed single-valued on B, and at a point of B common to two of the closed regions $\bar{R}_j$ the function $f(z)$ may not be single-valued. This phenomenon can occur in at most a finite number of points of B, and hence does not affect the validity of (15).

On each circle Γ_j we define a new function $f_1(z)$ equal to $f(z)$ on the part of Γ_j that belongs to B, and equal to zero elsewhere. We may write (15) in the form

$$f(z) = \frac{1}{2\pi i}\sum_{j=1}^{k}\int_{\Gamma_j} \frac{f_1(t)\,dt}{t - z}, \quad z \text{ interior to } R; \tag{16}$$

these integrals may still be taken in the sense of Riemann.

The function

$$\frac{1}{2\pi i}\int_{\Gamma_j} \frac{f_1(t)\,dt}{t - z}$$

is analytic throughout the interior of C_j and may be expressed there as the sum of a series

$$\sum_{n=0}^{\infty} a_{jn} (z - \alpha_j)^n \qquad \text{or} \qquad \sum_{n=0}^{\infty} a_{jn} (z - \alpha_j)^{-n},$$

where α_j is the center of Γ_j, according as C_j is interior or exterior to Γ_j. This series converges uniformly on any closed set interior to C_j. Hence we may write from (16)

$$f(z) = \sum_{n=0}^{\infty} \sum_{j=1}^{k} a_{jn}(z - \alpha_j)^{\pm n}, \quad z \text{ interior to } R,$$

where for each j the sign $\pm$ has only a single sense; the series converges uniformly on any closed set interior to R. The proof is complete.

In a given situation, if the original function $f(z)$ is not defined in *all* the regions R_j common to the C_j, it may be defined as identically zero in the remaining regions R_j. This is accomplished automatically by writing (15) in the form (16), if (16) is considered to define $f(z)$ in the new regions R_j, and this extended function $f(z)$ is represented by the development derived. Under such conditions the new function $f(z)$ may of course fail to be single-valued in certain points on B, points common to two of the $\bar{R}_j$, but the reasoning already given is essentially valid.

The method of Theorem 12 may be employed in the proof of Theorem 5, as we now indicate rapidly. If C' is an arbitrary closed set interior to Γ, there exist circles $C_1, C_2, \cdots, C_k$ with centers on Γ, with C' exterior to all these circles and with every point of Γ interior to some C_j; it is sufficient to cover Γ by circles with centers on Γ and with C' exterior to every such circle, and to apply the Heine-Borel theorem. The given function $f(z)$ is defined interior to Γ, and we extend the definition so that the new function is identically zero in the regions R_j exterior to Γ and exterior to and bounded by the C_j. The function

$$\frac{1}{2\pi i} \int_{C_j} \frac{f(t)\,dt}{t - z}$$

can be expanded exterior to C_j in a series

$$\sum_{n=0}^{\infty} a_{jn}(z - \alpha_j)^{-n},$$

where α_j is the center of C_j, valid uniformly on any closed set exterior to C_j. This process carried out for each C_j yields a rational function whose poles lie in the α_j and which approximates $f(z)$ on C'. The method obviously applies to the study of simultaneous approximation interior to several contours.

Still another method of approximation is expressed as follows [Wolff, 1921]:

THEOREM 13. *Let Γ be a limited Jordan region and let the function $f(z)$ be analytic in the corresponding closed region $\bar{\Gamma}$. Then the function $f(z)$ can be expressed interior to Γ in a series of the form*

$$f(z) = \sum_{k=1}^{\infty} \frac{A_k}{z - z_k}, \qquad \sum | A_k | \textit{ convergent}, \tag{17}$$

which converges to $f(z)$ interior to Γ, uniformly on any closed set interior to Γ.

Series of the form (17) have been much studied (for instance by Poincaré, Borel [1898], Carleman, Denjoy [1926], and others) and have important properties. Theorem 13 is not intended to replace Theorem 5, but rather to supplement it.

Let the sequence of contours $\Gamma_1, \Gamma_2, \cdots$ of respective lengths $l_1, l_2, \cdots$ lie exterior to Γ, contain Γ in their interiors, and approach Γ monotonically in the sense of the Corollary to Theorem 2. We suppose, as we may do, that $f(z)$ is analytic on and interior to every Γ_k. Let the positive numbers $\epsilon_1, \epsilon_2, \cdots$ be so chosen that the series $\sum l_k \epsilon_k$ converges; then ϵ_k approaches zero with $1/k$. Choose points $z_{11}, z_{12}, \cdots, z_{1n_1}$ on Γ_1 by the method of Theorem 5 so that we have

$$\left| f(z) - \frac{1}{2\pi i} \sum_{k=1}^{n_1} \frac{f(z_{1k})\, \Delta_{1k} z}{z - z_{1k}} \right| \leqq \epsilon_2, \quad z \text{ on } \Gamma_2.$$

Denote the function between vertical bars by $f_2(z)$, which is analytic on and within Γ_2, and choose points $z_{21}, z_{22}, \cdots, z_{2n_2}$ on Γ_2 so that we have

$$\left| f_2(z) - \frac{1}{2\pi i} \sum_{k=1}^{n_2} \frac{f_2(z_{2k})\, \Delta_{2k} z}{z - z_{2k}} \right| \leqq \epsilon_3, \quad z \text{ on } \Gamma_3 .$$

Denote the new function between vertical bars by $f_3(z)$, and continue this process. In general we choose points $z_{m1}, z_{m2}, \cdots, z_{mn_m}$ on Γ_m so that we have

$$\left| f_m(z) - \frac{1}{2\pi i} \sum_{k=1}^{n_m} \frac{f_m(z_{mk})\, \Delta_{mk} z}{z - z_{mk}} \right| \leqq \epsilon_{m+1}, \qquad z \text{ on } \Gamma_{m+1} .$$

We can write the general inequality in the form

$$\left| f(z) - \frac{1}{2\pi i} \sum_{m=1}^{n} \sum_{k=1}^{n_m} \frac{f_m(z_{mk})\, \Delta_{mk} z}{z - z_{mk}} \right| \leqq \epsilon_{n+1}, \qquad z \text{ on } \Gamma_{n+1}, \tag{18}$$

where we set $f_1(z) \equiv f(z)$. The series

$$\frac{1}{2\pi i} \sum_{m=1}^{\infty} \sum_{k=1}^{n_m} \frac{f_m(z_{mk})\, \Delta_{mk} z}{z - z_{mk}}$$

can be written in form (17), if the A_ν are chosen as the numbers $f_m(z_{mk})\Delta_{mk} z/(2\pi i)$ taken in the order suggested by this double summation, and if the z_ν are chosen as the z_{mk} in the corresponding order. The inequality

$$| f_m(z) | \leqq \epsilon_m, \qquad z \text{ on } \Gamma_m, \qquad m > 1,$$

yields

$$\sum_{k=1}^{n_m} |f_m(z_{mk})\Delta_{mk}z| \leqq \epsilon_m l_m ,$$

so the series $\sum |A_j|$ converges; the sum of series (17) is independent of order of summation for z interior to Γ.

Inequality (18) expresses the uniform convergence to $f(z)$ on any closed interior to Γ merely of a suitably chosen sequence of partial sums of series (but the conclusion of the theorem follows from the convergence of $\sum |A_j|$ the uniform boundedness on an arbitrary closed set interior to Γ of the qua ties $1/(z - z_j)$.

Series (17) converges and represents a function which is analytic also exte to Γ except in the points z_j, and which has Γ as a natural boundary except in trivial case that $f(z)$ vanishes identically.

§1.9. On the vanishing of analytic functions

We have thus far considered sufficient conditions for uniform approximat of an analytic function. In preparation for the study of necessary con tions we need certain preliminary results on the vanishing of analytic functio

THEOREM 14. *Let R be a simply connected region of the extended plane wh boundary B is not a single point, let the function $f(z)$ be single-valued and analy interior to R in the neighborhood of B, and let* $\lim_{z_k \to z_0} f(z_k)$ *exist and be eq to zero whenever the points z_k lie interior to R and approach a point z_0 of Then the function $f(z)$ vanishes identically interior to R in the neighborhood of*

By a *neighborhood of the boundary B* we understand here all the points of neighborhood of each point of B, with the omission of points not in R. It f lows from Theorem 4 and the method of Theorem 7 that this neighborhood of may be chosen as an annular region bounded by B and a Jordan curve interi to R. By a conformal map of R onto the interior of a circle, it is sufficient prove Theorem 14 for the case that R is the interior of the circle C: $|z| =$ The function $f(z)$ is then single-valued and analytic in some neighborhood of interior to C, and when z approaches C the function $f(z)$ approaches zero. T function can be assigned the value zero on the circumference C, and is then co tinuous in a closed neighborhood of C. The function $f(z)$ can be extend analytically across C by Schwarz's principle of reflection. The function $f_1(z$

$$f_1(z) \equiv f(z) , \qquad z \text{ on or interior to } C ,$$

$$f_1(z) \equiv \overline{f(1/\bar{z})} , \qquad z \text{ exterior to } C ,$$

is single-valued and analytic in an annular region which contains C in its interio The function $f_1(z)$ vanishes on C, hence vanishes identically throughout th annular region.

A somewhat more general theorem is to be used in the sequel:

THEOREM 15. *Let R be an arbitrary region of finite connectivity of the extended plane whose boundary B has no isolated points. Let the function $f(z)$ be single-valued and analytic in R in the neighborhood of B, and let $\lim_{z_k \to z_0} f(z_k)$ exist and be equal to zero whenever the points z_k lie interior to R and approach a point z_0 of B. Then the function $f(z)$ vanishes identically interior to R in the neighborhood of B.*

It is clear that the function may be defined on some closed point set interior to R, not be identically zero there, and still satisfy the conditions of the theorem. Moreover, it is clear that the restriction that B should have no isolated points is essential, for a function $f(z)$ may be analytic in the neighborhood of such a point and approach zero whenever z approaches that point without vanishing identically in that neighborhood.

Let $K_1, K_2, \cdots, K_\nu$ denote the components of the complement of R. The complement of K_m is a simply connected region R_m which contains in its interior every (interior) point of R. The function $f(z)$ is single-valued and analytic in the neighborhood of the boundary of R_m and approaches zero whenever z interior to R_m approaches a point of K_m. It follows from Theorem 14 that $f(z)$ vanishes identically in the neighborhood of the boundary of R_m; this is true for every m, so the proof is complete.

Theorems 14 and 15 can be somewhat extended by the reasoning already used, for it is not essential in the proof of Theorem 14 that the region of definition of $f(z)$ be a neighborhood of B; it is sufficient if that region of definition is a neighborhood of a part of B which under the conformal mapping of R onto the interior of a circle C corresponds to an arc of C. But the more general reasoning thus suggested does not allow us to omit in Theorem 15 the requirement of finite connectivity of R, if R is of infinite connectivity, a given component K_m of the complement of R may possess no accessible boundary points of R; the function $f(z)$ need not be defined throughout a neighborhood interior to R_m of a part of K_m which under the conformal map of R_m onto the interior of a circle corresponds to an arc of that circle. For the sake of simplicity, we shall limit ourselves in the applications of Theorems 14 and 15 to cases where the formulated theorems suffice.

§1.10. Necessary conditions for approximation

We shall now study necessary conditions for the approximation of analytic functions by rational functions, and shall assume that the given function is analytic on the entire closed set on which approximation is studied; the case that the given function satisfies weaker conditions is more delicate and will be discussed to some extent in Chapter II. For simplicity, we discuss only the case that points z_k (not necessarily denumerable) are given, in which the poles of the approximating rational functions are allowed to lie. Only obvious modifications are necessary to include the case that the poles are prescribed by a given array as at the end of §1.6.

It will be sufficient in the future for us to restrict our study to approximation

on a closed point set of the extended plane, for as we shall now prove, *appro-* *mation on a set C which is not closed can be reduced to approximation on the se* *composed of C and its limit points.* Instead of considering approximation, it more convenient here to use the equivalent property of uniform expansion. I the sequence of rational functions $r_n(z)$ converge uniformly to $f(z)$ on C, $\epsilon > 0$ is arbitrary, there exists N such that the inequalities $m > N, n > N$ im

$$(19) \qquad |r_n(z) - r_m(z)| \leqq \epsilon, \qquad z \text{ on } C\,.$$

Since this inequality holds for z on C, it holds also for z on $\bar{C}$, and therefore t sequence $r_n(z)$ converges uniformly on $\bar{C}$. That is, the function $f(z)$ defin merely on C, can also be defined continuously and uniformly approximated on

In the discussion just given, the function $f(z)$ can be defined continuously $\bar{C}$ *provided the infinite constant is admitted to the number system*, contrary to usual convention. It can easily occur that all of the $r_n(z)$ have a common pol on $\bar{C}$, so inequalities (19) and

$$(20) \qquad |f(z) - r_n(z)| \leqq \epsilon, \qquad z \text{ on } \bar{C}\,,$$

hold at α only in the sense of approach of z to α. Let us now indicate h *approximation on a closed set $\bar{C}$ of the extended plane, where the approximated c* *tinuous function $f(z)$ is allowed to assume the value infinity (although not identical* *at non-isolated points of $\bar{C}$, can be reduced to approximation of a continuous boun* *function on that set.*

If the function $f(z)$ is approximated on $\bar{C}$, it can assume the value infinity in most a finite number of points, by (20), for $r_n(z)$ can become infinite in at mos finite number of points of $\bar{C}$. Let α be an arbitrary one of these points. If functions $r_n(z)$ approach $f(z)$ uniformly on $\bar{C}$, inequality (19) is valid on $\bar{C}$ un the conditions given, so all the functions $r_n(z)$ for n sufficiently large have same principal part $p_\alpha(z)$ at α. Each of the functions $r_n(z) - p_\alpha(z)$ is conti ous and bounded in some neighborhood of α, when suitably defined for $z =$ and all of those functions are continuous and uniformly bounded on $\bar{C}$ at points of a neighborhood of α which does not depend on n. Hence the quence of rational functions $r_n(z) - p_\alpha(z)$ converges uniformly to $f(z) - p$ on the points of $\bar{C}$ in this neighborhood of α, and the latter function is conti ous and bounded on these points, including the point α itself if a suitable d nition is supplied there. This reasoning can be applied successively at eacl the finite number of points α of $\bar{C}$ where $f(z) = \infty$. Thus, a necessary co tion that $f(z)$ can be approximated on $\bar{C}$ is that the continuous bounded fu tion $f(z) - \sum_\alpha p_\alpha(z)$ can be approximated on $\bar{C}$ by rational functions whose p do not lie on $\bar{C}$. This condition is obviously sufficient, so the given proble reduced to the problem of approximating a continuous bounded function o The function $p_\alpha(z)$ can be determined from a knowledge of $f(z)$ and $\bar{C}$ al whether or not it is known that $f(z)$ can be approximated. There must e a function $p_\alpha(z)$ of the form $\sum_{k=1}^{m} a_k(z - \alpha)^{-k}$ (an obvious alteration is to made if α is the point at infinity) such that $f(z) - p_\alpha(z)$ is bounded in neighborhood of α.

A necessary and sufficient condition that a function (not necessarily bounded) can be uniformly approximated on a closed set $\bar{C}$ by rational functions whose poles lie on $\bar{C}$ is that the function be a rational function whose poles lie on $\bar{C}$. The proof of a particular case is typical: *a necessary and sufficient condition that a function $f(z)$ can be uniformly approximated on an unlimited point set C by a polynomial in z is that $f(z)$ itself should be a polynomial in z.* If polynomials $p_n(z)$ approach $f(z)$ uniformly on C, all those polynomials for n sufficiently large must have the same principal part $p(z)$ at infinity, so $p_n(z) - p(z)$ is a constant and therefore approaches a constant on C. Thus $f(z) - p(z)$ is a constant on C and $f(z)$ is a polynomial.

In the sequel, then, it is sufficient for us to restrict ourselves to the study of approximation to a continuous (and bounded) function on a closed point set which contains none of the assigned points z_k, possible poles of the approximating functions. We proceed to prove the following theorem containing our main result on necessary conditions for approximation.

THEOREM 16. *Let C be a closed point set of the extended plane, and let the points z_k (not necessarily denumerable) lie exterior to C and in only a finite number of the regions into which C separates the plane. A function $f(z)$ analytic on C can be uniformly approximated on C by a rational function whose poles lie in the points z_k if and only if the function $f(z)$ can be extended analytically from C so as to be single-valued and analytic in every point of the plane which is separated by C from the points z_k. That is to say, this condition is that there should exist a function single-valued and analytic not merely on C but also in every point of the plane separated by C from the points z_k, and which coincides with $f(z)$ on C.*

In particular, a function $f(z)$ analytic on a closed limited point set C can be uniformly approximated on C by a polynomial in z if and only if the function $f(z)$ can be extended analytically from C so as to be single-valued and analytic in every point of the plane which is separated by C from the point at infinity.

The first part of the theorem is a direct consequence of Theorem 8; the function $f(z)$ can be approximated not merely on C, but on the set consisting of C and the points of the plane separated by C from the points z_k. We proceed to prove the second part.

Let $r_n(z)$ be a sequence of rational functions whose poles lie in the prescribed points z_k and which converges to $f(z)$ uniformly on C. Let R' be any one of the regions into which C separates the plane which is also separated by C from the points z_k. Let R be the region formed by enlarging R' by suppressing the components (if any) of the boundary of R' which do not effectively separate R' from points z_k, so that R contains every point of R' in its interior. The boundary B of R can have no isolated points, every point of B is a point of C, the region R is of finite connectivity, and B separates every point of R from the points z_k.

The sequence $r_n(z)$ converges uniformly to $f(z)$ on C, hence converges uniformly in the closed region $\bar{R}$ to some function $F(z)$, which is analytic in R and continuous in $\bar{R}$, and coincides with $f(z)$ on B. The function $f(z)$ is analytic on

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In particular, a function $f(z)$ analytic on a closed limited point set C can be uniformly approximated on C by a polynomial in z if and only if the function $f(z)$ can be extended analytically from C so as to be single-valued and analytic in every point of the plane which is separated by C from the point at infinity.

The first part of the theorem is a direct consequence of Theorem 8; the function $f(z)$ can be approximated not merely on C, but on the set consisting of C and the points of the plane separated by C from the points z_k. We proceed to prove the second part.

Let $r_n(z)$ be a sequence of rational functions whose poles lie in the prescribed points z_k and which converges to $f(z)$ uniformly on C. Let R' be any one of the regions into which C separates the plane which is also separated by C from the points z_k. Let R be the region formed by enlarging R' by suppressing the components (if any) of the boundary of R' which do not effectively separate R' from points z_k, so that R contains every point of R' in its interior. The boundary B of R can have no isolated points, every point of B is a point of C, the region R is of finite connectivity, and B separates every point of R from the points z_k.

The sequence $r_n(z)$ converges uniformly to $f(z)$ on C, hence converges uniformly in the closed region $\bar{R}$ to some function $F(z)$, which is analytic in R and continuous in $\bar{R}$, and coincides with $f(z)$ on B. The function $f(z)$ is analytic on

C, hence can be extended analytically from C into R in the neighborhood of B. The function $F(z) - f(z)$ is analytic in the neighborhood of B and approaches zero whenever z interior to R approaches B. It follows from Theorem 15 that $F(z) - f(z)$ vanishes identically in the neighborhood of B, so the analytic extension of $f(z)$ from B interior to R coincides with $F(z)$. This is true for every region R, so the proof is complete. Indeed, we have shown that $f(z)$ can be extended from C so as to be single-valued and analytic not merely in every region R' but in every region R.

Theorem 16 is of especial interest when C is a closed region; compare §1.6.

From Theorem 16 follows without difficulty

THEOREM 17. *Let C be an arbitrary closed point set of the extended plane, and let points z_k (not necessarily denumerable) be given not belonging to C but lying in only a finite number of the regions into which C separates the plane. A necessary and sufficient condition that EVERY function $f(z)$ analytic on C can be uniformly approximated on C by a rational function whose poles lie in the points z_k is that at least one point z_k lie in each of the regions into which C separates the plane.*

In particular, a necessary and sufficient condition that every function analytic on a closed limited point set C can be uniformly approximated on C by a polynomial is that C should not separate the plane, or (the equivalent) that C should be the complement of an infinite region.

Only the necessity of this condition remains to be proved. If the set C does not satisfy the given condition, let R be one of the regions into which C separates the plane and which contains no point z_k. Let $z = \alpha$ be a point of R. It follows from Theorem 16 that the function $f(z) \equiv 1/(z - \alpha)$, which is analytic on C, cannot be uniformly approximated on C by a rational function whose poles lie in the points z_k.

Theorems 14–17, so far as they apply to approximation by polynomials, are given by Walsh [1926, 1926a, 1928a, 1929b]; compare also, especially for *non-uniform* expansion by polynomials and rational functions respectively, Hartogs and Rosenthal [1928], and Carathéodory [1928].

The question of approximation on a closed point set C of a function not known to be analytic on C is much more delicate than the contrary case. Certain sufficient conditions for approximation can be obtained by the use of the modern theory of conformal mapping, and we shall develop such conditions in the next chapter.

CHAPTER II

POSSIBILITY OF APPROXIMATION, CONTINUED

§2.1. Lindelöf's first theorem

There are a number of results in the modern theory of conformal mapping that we shall need later, and which we shall proceed to prove in detail. The first of these results is due to Lindelöf [1915]:

THEOREM 1. *Let T be a limited simply connected region of the z-plane which contains the origin O in its interior. Let Δ be the greatest diameter of T and let δ be the distance from O to the boundary of T. When T is mapped conformally onto the interior of $\gamma : |w| = 1$ so that the origins in the two planes correspond to each other, then every point of T at a distance less than $r(< \delta)$ from the boundary of T corresponds to a point w whose distance from γ is less than*

$$\delta(r) = \frac{2\log\Delta - 2\log\delta}{2\log\Delta - \log(\delta r)};$$

the expression $\delta(r)$ approaches zero with r.

Let $z = a$ be an arbitrary point on the boundary of T, construct the Riemann surface with an infinity of sheets corresponding to the function $\log(z - a)$, and denote by T' the part of that surface interior to the circle $|z - a| < \Delta$. If $z = 0$ is associated with a particular sheet of the surface, the region T can be considered as belonging to the Riemann surface.

The region T' is mapped onto the half-plane* $\Re(t) < \log\Delta$ by the transformation $z - a = e^t$; the point $z = 0$ corresponds to the point $t = t_0 = \log A + i\alpha$, where $a = -Ae^{i\alpha}$, $A > 0$. Let us now map the half-plane $\Re(t) < \log\Delta$ onto the interior of the unit circle $\gamma : |w| = 1$ so that the point $t = t_0$ is transformed into $w = 0$; it is sufficient to set

$$w = \frac{t - t_0}{t - t_1},$$

where t_1 is the reflection of t_0 in the line $\Re(t) = \log\Delta$; we have $t_1 = 2\log\Delta - \log A + i\alpha$. The corresponding transformation of T itself is given by

$$w = F(z) \equiv \frac{\log(z - a) - \log A - i\alpha}{\log(z - a) - 2\log\Delta + \log A - i\alpha}.$$

The part of the region T' interior to the circle $|z - a| < r < \delta$ corresponds in the t-plane to the region $\Re(t) < \log r$, which corresponds in the w-plane to the interior of the circle $|w - 1 + \delta'| = \delta'$,

* The symbol $\Re(t)$ means *the real part of t*.

$$\delta' = \frac{\log \Delta - \log A}{2 \log \Delta - \log(Ar)};$$

the point $t = \log r + i\alpha$ corresponds to $w = 1 - 2\delta'$. It follows then that the inequality $| z - a | < r$ implies $| F(z) | > 1 - 2\delta'$.

Let $w = \Phi(z)$ map the interior of T onto the interior of γ so that the points $z = 0, w = 0$ correspond to each other. Then the function

$$\frac{F(z)}{\Phi(z)}$$

is analytic at every point interior to T when suitably defined for $z = 0$. Whenever a sequence of points z interior to T approaches the boundary of T, the function $| \Phi(z) |$ approaches unity and the function $| F(z) |$ has no superior limit greater than unity. Thus we have (Principle of Maximum)

$$\text{(1)} \qquad \left| \frac{F(z)}{\Phi(z)} \right| < 1, \qquad \text{or} \qquad | F(z) | < | \Phi(z) | \qquad (z \neq 0),$$

for z interior to T; the equality could hold here at a point $z (\neq 0)$ interior to T only if $F(z)$ and $\Phi(z)$ were identical except for a constant factor of modulus unity, which we know to be not the case.

At a point z interior to T and interior to a circle $| z - a | < r$, we have by virtue of (1) the inequality $| \Phi(z) | > 1 - 2\delta'$. When the point a varies, δ' also varies but never exceeds $\delta(r)/2$, for we have $A \geqq \delta$; it may be verified (by differentiation or otherwise) that δ' decreases as A increases. The proof is complete.

Theorem 1 has some implications which we shall use frequently in Chapter V, so we turn to their proof before proceeding with the development of the present chapter.

THEOREM 2. *Let T be a limited simply connected region of the z-plane and let T' be a variable Jordan region which together with its boundary lies interior to T. Let $w = \phi(z)$ map the exterior of T' onto the exterior of γ: $| w | = 1$ so that the points at infinity correspond to each other, and let T'_R denote the curve $| \phi(z) | = R > 1$ exterior to T'. If $R > 1$ is given, then T' (interior to T) can be chosen so that the closed region $\overline{T}$ lies interior to T'_R.*

The region T need not be a Jordan region, its boundary need not be the boundary of an infinite region, and its boundary may separate the plane into more than two regions.

Consider a monotonic sequence of regions T' interior to T, such as occurs in §1.3, Theorem 2. Let the point $O: z = 0$ be common to all regions of the sequence. A situation similar to that of Theorem 1 is found by making the transformations $z_1 = 1/z$, $w_1 = 1/w$. The region T is transformed into a region whose complement is denoted by T_1, the closed region $\overline{T'}$ is transformed into a closed region whose complement is denoted by T'_1, the exterior of γ is transformed into the interior of γ_1: $| w_1 | = 1$, and the region $1 < | w | < R$ into the region $1/R < | w_1 | < 1$. We are now in a position to apply Theorem 1. When the region T'_1 varies and its boundary approaches the boundary of T_1, the maximum diameter of T'_1

approaches the maximum diameter of T_1, and the distance from the point $z_1 = 0$ to the boundary of T_1' approaches the distance from the point $z_1 = 0$ to the boundary of T_1. If the region T_1' is so chosen that every point of the boundary of T_1 is within a suitably small distance r of the boundary of T_1', which is possible by §1.3, Theorem 4, it follows from Theorem 1 that when T_1' is mapped onto the interior of γ_1, the boundary of T_1 corresponds in the w_1-plane to a point set in the ring $1/R < | w_1 | < 1$. Then $\overline{T}$ lies interior to T_R', and the proof is complete.

Theorem 2 follows not merely as here from Theorem 1, due to Lindelöf, but can also be proved from well known results on the mapping of variable regions due to Carathéodory [1912]; compare Corollary 2 to Theorem 4 below.

If T in Theorem 2 is chosen not as a region but as a Jordan arc, and if T' is a variable subarc, the point set T_R' may be defined as before, but the proof of the conclusion requires some modification. Let us indicate the proof of the

COROLLARY. *Let T be a limited Jordan arc and let T' be a variable subarc of T whose end-points are both different from the end-points of T. Let $w = \phi(z)$ map the complement of T' onto the exterior of $\gamma : | w | = 1$ so that the points at infinity correspond to each other, and let T_R' denote the curve $| \phi(z) | = R > 1$. If $R > 1$ is given, then T' can be chosen so that T lies interior to T_R'.*

Let $z = \alpha$ and $z = \beta$ be two points of T', different from the end-points of T'. The transformation

$$\frac{\tau - \alpha}{\tau - \beta} = \left(\frac{z - \alpha}{z - \beta}\right)^{1/2}$$

maps the complements of T and T' onto simply connected regions S and S' of the τ-plane for which *exterior points* exist. Even if S' is not now a Jordan region, it is entirely possible to make the transformations $z' = 1/\tau$, $w_1 = 1/w$, where $\tau = 0$ is any point exterior to S', and to proceed as in the proof of Theorem 2.

The corollary can also be extended to apply to various point sets more general than Jordan arcs

§2.2. Lindelöf's second theorem

We shall now establish an important complement of Theorem 1, likewise due to Lindelöf [1915]:

THEOREM 3. *Under the hypothesis of Theorem 1, let a be an arbitrary point of the boundary of T, let l be a Jordan arc in T contained in the circle $| z - a | < r < \delta$, and let σ be the oscillation on l of the argument of the mapping function $\Phi(z)$; then for r sufficiently small we have*

$$\sigma < 4 \tan^{-1} \left(\frac{2 \log \Delta - 2 \log \delta}{\log \Delta - \log r}\right)^{1/2},$$

a quantity which approaches zero with r.

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We suppose, as we may do with no loss of generality, that the greatest and least values θ_1 and θ_2 which the argument θ of $\Phi(z)$ assumes on the arc l are assumed in the end-points of l. That is to say, the transform in the w-plane of the Jordan arc l is a Jordan arc λ whose extremities are points A_1 and B_1 on the respective radii $\theta = \theta_1$ and θ_2, while the remainder of λ lies in the sector $\theta_2 < \theta < \theta_1$. Let A and B denote the respective points where the radii OA_1 and OB_1 meet $\gamma : | w | = 1$.

w-plane

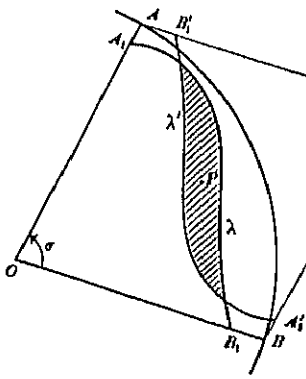

FIG. 1

By Theorem 1, the arc λ lies between γ and the concentric circle γ_1 whose radius is $1 - \delta(r)$. A tangent to γ_1 intercepts on γ an arc of length

$$\sigma_1 = 2 \cos^{-1} [1 - \delta(r)] .$$

For the present it is convenient to make two restrictions on the difference $\sigma = \theta_1 - \theta_2$:

$$\sigma_1 < \sigma \leqq \pi/2 ; \tag{2}$$

we shall later discuss the significance of (2).

Let $P: w = w_0$ denote the mid-point of AB; then by (2) the point P lies interior to γ_1. Hence P lies in the region Ω bounded by λ and the segments OA_1 and OB_1. Rotate Ω about P through an angle π, and denote by Ω', λ', O', A_1', B_1' the respective transforms of Ω, λ, O, A_1, B_1. Since σ is not greater than $\pi/2$, the segments $O'A_1'$ and $O'B_1'$ are exterior to γ and to Ω; likewise OA_1 and OB_1 are exterior to Ω'. The point P lies in both Ω and Ω', hence lies in some region Ω_0 common to Ω and Ω', bounded only by arcs of λ and λ'.

Denote by $z = \Psi(w)$ the inverse of the function $w = \Phi(z)$, and let us set

$$\Psi_0(w) = \Psi(w) - a .$$

This function is analytic in the closed region $\bar{\Omega}$, satisfies the inequality $| \Psi_0(w) | < \Delta$ in $\bar{\Omega}$, and satisfies the more restrictive inequality $| \Psi_0(w) | < r$ on λ. Under the

rotation about P used above, $w' = 2w_0 - w$, the function $\Psi_0(w)$ is transformed into the function $\Psi_1(w) \equiv \Psi_0(2w_0 - w)$ analytic in the closed region $\bar{\Omega}'$. We have $|\Psi_1(w)| < \Delta$ in $\bar{\Omega}'$, $|\Psi_1(w)| < r$ on λ'.

Thus the function $\Psi_0(w)\cdot\Psi_1(w)$ is analytic in the closed region $\bar{\Omega}_0$, and on the boundary satisfies the inequality $|\Psi_0(w)\cdot\Psi_1(w)| < \Delta r$. Consequently this inequality is valid at every interior point of Ω_0; in particular for $w = w_0$ we have

$$|\Psi(w_0) - a| < (\Delta r)^{1/2}.$$

It follows then that the distance from the boundary of T of the point $z = \Psi(w_0)$ is less than $(\Delta r)^{1/2}$. The latter quantity is less than δ provided we have $r < \delta^2/\Delta$. For such values of r the distance of P from γ is by Theorem 1 less than

$$\delta[(\Delta r)^{1/2}] = \frac{4 \log \Delta - 4 \log \delta}{3 \log \Delta - 2 \log \delta - \log r}. \tag{3}$$

On the other hand, the distance of P from γ is

$$1 - \cos(\sigma/2) = 2 \sin^2(\sigma/4),$$

so we have

$$2 \sin^2(\sigma/4) < \frac{4 \log \Delta - 4 \log \delta}{3 \log \Delta - 2 \log \delta - \log r},$$

$$\sigma < 4 \sin^{-1}\left(\frac{2 \log \Delta - 2 \log \delta}{3 \log \Delta - 2 \log \delta - \log r}\right)^{1/2} = 4 \tan^{-1}\left(\frac{2 \log \Delta - 2 \log \delta}{\log \Delta - \log r}\right)^{1/2}. \tag{4}$$

This is the inequality desired, thus far established for $r < \delta^2/\Delta$, provided (2) is valid.

The requirement $\sigma > \sigma_1$ is now unnecessary, for the extreme right-hand member of (4) can be expressed as

$$2 \cos^{-1}\{1 - \delta[(\Delta r)^{1/2}]\},$$

which is itself greater than σ_1. That is to say, the theorem asserts that σ is less than some quantity which is itself greater than σ_1; if σ is already less than or equal to σ_1, there is nothing further to be proved

The requirement $\sigma \leqq \pi/2$ is a formal consequence of (4) provided r is sufficiently small; it is sufficient to take r less than

$$r_0 = \delta(\delta/\Delta)^{5+4(2)^{1/2}};$$

this follows from a study of the functions which appear in (4); for $r = r_0$, inequality (4) becomes an *equality*. This exact expression for r_0 is indeed a matter of no consequence in the proof of the theorem; it is sufficient to notice that $\sigma \leqq \pi/2$ is a consequence of (4) when r is less than some r_0, where r_0 depends only on δ and Δ. The new inequality $r < r_0$ is more restrictive than the inequality $r < \delta^2/\Delta$ previously introduced, so this previous inequality is satisfied automatically if r is less than r_0.

If now we have $r < r_0$, the inequality $\sigma > \pi/2$ cannot occur. In the contrary case, there would exist a subarc of the given Jordan arc l on which the oscillation of $\Phi(z)$ would be less than or equal to $\pi/2$ and still greater than the limit given in (4); this has been proved to be impossible. Thus for $r < r_0$ inequality (4) is satisfied, and the theorem is completely proved.

Theorems 1 and 3 apply not merely to Jordan regions, but to arbitrary limited simply connected regions. They can be used, and are used by Lindelöf, to establish the continuity in the closed region of the mapping function for a Jordan region.

§2.3. Conformal mapping of variable regions

We shall apply Theorems 1 and 3 in the proof of a theorem of some importance for the sequel.

THEOREM 4. *Let T be a Jordan region of the finite z-plane and let the limited simply connected regions $T_1, T_2, \cdots$ all contain the closed region $\overline{T}$, let T_n contain $\overline{T}_{n+1}$ in its interior, and let no point exterior to T lie in all the T_n. Let the origin $z = 0$ lie interior to T, and let the respective functions $w = \Phi(z)$, $\Phi_n(z)$ map the regions T, T_n onto the interior of $\gamma : | w | = 1$ so that the point $z = 0$ corresponds to the point $w = 0$ and so that $\Phi'(0) > 0$, $\Phi_n'(0) > 0$. Then we have uniformly for z in $\overline{T}$,*

$$\lim_{n\to\infty} \Phi_n(z) = \Phi(z) . \tag{5}$$

As a matter of convenience, we introduce two lemmas.

LEMMA I. *Let the simply connected region S_1 of the z-plane be a proper subregion of the simply connected region S_2, and let the respective functions $w = F_1(z)$, $F_2(z)$ map S_1 and S_2 onto the interior of γ so that the point $z = 0$ (assumed interior to S_1) corresponds to the point $w = 0$, and so that $F_1'(0) > 0$, $F_2'(0) > 0$. Then we have*

$$F_1'(0) > F_2'(0) . \tag{6}$$

The function

$$F(z) \equiv \frac{F_1(z)}{F_2(z)}$$

is analytic and different from zero at every point of S_1, when suitably defined for $z = 0$. When z interior to S_1 approaches a boundary point z_0 of S_1, the modulus $| F_1(z) |$ approaches unity and the modulus $| F_2(z) |$ approaches values not greater than unity; the modulus $| F_2(z) |$ approaches values actually less than unity if z approaches boundary points z_0 of S_1 which are interior to S_2; such boundary points exist. Thus we have for z interior to S_1,

$$\left| \frac{F_1(z)}{F_2(z)} \right| > 1 .$$

If in particular we set $z = 0$, we obtain (6).

LEMMA II. *Under the hypothesis of Theorem 4, we have (5) valid at every point z interior to T, uniformly on any closed set interior to T.*

The inequality $| \Phi_n(z) | < 1$ is valid at every point interior to T, so interior to T the functions $\Phi_n(z)$ form a normal family. By Lemma I we have

$$\Phi_n'(0) < \Phi_{n+1}'(0) < \Phi'(0), \tag{7}$$

so the positive numbers $\Phi_n'(0)$ are monotonically increasing, and no limit function of the family can be identically constant. Each function $\Phi_n(z)$ is univalent (schlicht) interior to T, so by Hurwitz's theorem the same is true of every limit function of the set $\Phi_n(z)$. It remains merely to show that interior to T any limit function $\Phi_0(z)$ of the set $\Phi_n(z)$ coincides with $\Phi(z)$.

The function $\Phi_0(z)$ maps T onto some simply connected region γ_0 interior to γ, for every value taken on by $\Phi_n(z)$ is in modulus less than unity, so every value taken on by $\Phi_0(z)$ is in modulus less than unity; equality is excluded by Hurwitz's theorem and the Principle of Maximum. The point $w = 0$ lies interior to γ_0, for we have

$$\Phi_0(0) = \lim \Phi_{n_k}(0) = 0, \qquad \Phi_0'(0) = \lim \Phi_{n_k}'(0) > 0. \tag{8}$$

It is sufficient to show that γ_0 has no boundary point interior to γ. Let $\eta > 0$ be arbitrary, and let T_0 denote an annular region interior to T bounded by the boundary of T and by a Jordan curve interior to T, such that every point of T_0 is within a distance η of the boundary of T. Let N be chosen so that every boundary point of T is within a distance η of the boundary of T_N; every boundary point of T is also within a distance η of the boundary of T_n, $n > N$, and every point of T_0 is within a distance 2η of the boundary of T_n, $n > N$. For n sufficiently large, the function $w = \Phi_n(z)$ maps T_0 onto a region whose maximum distance from γ is less than $\delta_n(2\eta)$, this notation is the notation of Theorem 1, where the constants Δ and δ refer to the region T_n. Moreover, we have (notation of Theorem 1) $\delta_n(2\eta) < \delta_0(2\eta)$ uniformly with respect to n, where $\delta_0(2\eta)$ is suitably chosen and independent of n and approaches zero with η; indeed, the constants Δ and δ for the region T_n approach the corresponding constants for the region T. That is to say, for z in T_0 we have

$$| \Phi_n(z) | > 1 - \delta_0(2\eta),$$

and hence for z in T_0 we also have

$$| \Phi_0(z) | \geqq 1 - \delta_0(2\eta).$$

Consequently, each point of the boundary of γ_0 lies on or exterior to the circle $| w | = 1 - \delta_0(2\eta)$, where η is merely sufficiently small. Then γ_0 has no boundary point interior to γ, so γ_0 is the interior of γ, and Lemma II is established.

We are now in a position to prove Theorem 4. A rapid indication of the method of proof is the following. Equation (5) is valid uniformly for z on any closed point set T' interior to T. If T' is chosen sufficiently large, the difference

between the value of $\Phi_n(z)$ at a point of $\overline{T}$ not of T' and the value of $\Phi_n(z)$ at a suitably chosen point of T' is arbitrarily small, uniformly with respect to n and z; similarly, the difference between the value of $\Phi(z)$ at a point of $\overline{T}$ not of T' and the value of $\Phi(z)$ at this suitably chosen point of T' is arbitrarily small uniformly with respect to z. This implies equation (5) uniformly in $\overline{T}$. Let us make this proof more precise.

Let an arbitrary positive ϵ be given. Theorems 1 and 3 indicate restrictions on the oscillation of the modulus and argument respectively of $\Phi(z)$ and $\Phi_n(z)$ on Jordan arcs interior to T and T_n. There exists $r > 0$ independent of n such that on any Jordan arc interior to T or T_n respectively which lies entirely within a distance r of a boundary point a of T or T_n the maximum oscillation of $\Phi(z)$ or $\Phi_n(z)$ is less than $\epsilon/3$; maximum oscillation say of $\Phi(z)$ means here the maximum of $| \Phi(z_1) - \Phi(z_2) |$ for z_1 and z_2 on the Jordan arc. By the continuity properties of the functions $\Phi(z)$ and $\Phi_n(z)$ in $\overline{T}$ and $\overline{T}_n$, it is allowable here to have an end-point of such a Jordan arc a boundary point of T or T_n.

Choose the closed point set T' interior to T such that every point z belonging to $\overline{T}$ but not to T' can be joined to a suitably chosen point z' (depending on z) of T' by a Jordan arc J which lies interior to T (except that an end-point of J is a boundary point of T if z itself is a boundary point of T), and such that J lies entirely within a distance $r/2$ of some boundary point a (depending on z) of T. It is possible to choose T' so that this condition is satisfied. Indeed, the corresponding problem can obviously be solved if we interpret the problem for the interior of the unit circle γ instead of for the region T; the desired closed set interior to γ may be chosen as the closed interior of a suitably chosen circle concentric with γ, and the arcs J may be chosen as segments of lines which pass through the origin. Thanks to the uniform continuity of the function which maps the interior of γ onto T, the solution of the auxiliary problem in γ leads to the solution of the original problem in T.

Let us now choose N such that every point of the boundary of T is within a distance $r/2$ of the boundary of T_N; this is possible (§1.3, Theorem 4) because the set of points at a distance from T not less than $r/2$ is closed; then every boundary point of T is within a distance $r/2$ of the boundary of T_n, $n > N$. Choose N also such that for $n \geqq N$ and for z' in T' we have (Lemma II) uniformly

$$(9) \qquad | \Phi(z') - \Phi_n(z') | < \epsilon/3 .$$

Let z be an arbitrary point of $\overline{T}$ but not a point of T', let J be the Jordan arc introduced above, and a and z' the corresponding points introduced. The point a on the boundary of T is within a distance $r/2$ of some point z_n (depending on a) on the boundary of $T_n (n \geqq N)$. Then the entire arc J lies within a distance r of this point z_n of the boundary of T_n. The maximum oscillation of $\Phi(z)$ and $\Phi_n(z)$ on J is less than $\epsilon/3$, so we have $(n \geqq N)$

$$| \Phi(z) - \Phi(z') | < \epsilon/3 , \qquad | \Phi_n(z) - \Phi_n(z') | < \epsilon/3 .$$

These inequalities together with (9) yield

$$|\Phi(z) - \Phi_n(z)| < \epsilon \qquad (n \geqq N), \tag{10}$$

for z in $\overline{T}$ not in T'; inequality (9) implies (10) for z in T'; the proof of Theorem 4 is complete.

Theorem 4 is similar to a theorem (in which the T_n are restricted to be Jordan regions) proved by Carathéodory and Courant [see Courant, 1914] by a somewhat different method.

COROLLARY 1. *Under the hypothesis of Theorem 4, let $z' = \chi_n(z)$ map the region T_n of the z-plane onto the region T of the z'-plane, $\chi_n(0) = 0$, $\chi_n'(0) > 0$. Then we have uniformly in $\overline{T}$*

$$\lim_{n\to\infty} \chi_n(z) = z. \tag{11}$$

Let $z = \Psi(w)$ be the inverse of the mapping function $w = \Phi(z)$. The function $w = \Phi_n(z)$ maps the region T_n of the z-plane onto the interior of γ, and the function $z' = \Psi(w)$ maps the interior of γ onto the interior of the region T of the z'-plane, so we can write

$$\chi_n(z) \equiv \Psi[\Phi_n(z)].$$

We naturally have

$$\Psi[\Phi(z)] \equiv z.$$

That is to say, equation (11) is precisely the result of operating on both members of equation (5) by the function $\Psi(w)$, continuous in the *closed* interior of γ, so (11) is valid uniformly in $\overline{T}$.

The proof of the first part of the following corollary is precisely the same as the proof of Lemma II; the proof of the second part is the same as the proof of Corollary 1.

COROLLARY 2. *Let T be a limited simply connected region of the z-plane containing the origin in its interior and whose boundary is also the boundary of an infinite region K. Let $T_1, T_2, T_3, \cdots$ be a sequence of regions containing the closed region $\overline{T}$ whose boundaries lie in K. Let the sequence T_n approach the boundary of T monotonically in the sense that the closed region $\overline{T}_{n+1}$ lies interior to T_n and that every point of K lies exterior to some T_n. Let $w = \Phi(z)$, $z = \Psi(w)$, map the interior of T onto the interior of $\gamma : |w| = 1$ so that $\Phi(0) = 0$, $\Phi'(0) > 0$; let $w = \Phi_n(z)$ map the interior of T_n onto the interior of γ so that $\Phi_n(0) = 0$, $\Phi_n'(0) > 0$. Then we have (5) uniformly on any closed set interior to T.*

The function $z' = \chi_n(z) \equiv \Psi[\Phi_n(z)]$ maps the region T_n of the z-plane onto the region T of the z'-plane, and equation (11) is valid uniformly on any closed set interior to T.

Corollary 2 is also a consequence of the theory of conformal mapping of variable regions developed by Carathéodory [1912].

It is to be noted that the region K is not necessarily the complement of the closed region $\bar{T}$. For instance, T may be a strip closed at one end, winding infinitely many times around a circle from the exterior, and approaching that circle. The boundary of T (of which the circle is a part) is also the boundary of an infinite region.

§2.4. Approximation in a closed Jordan region

We are now in a position to obtain results on approximation by polynomials.

THEOREM 5. *Let C be a finite Jordan region of the z-plane. If the function $f(z)$ is analytic in C, continuous in the closed region $\bar{C}$, then in $\bar{C}$ the function $f(z)$ can be uniformly approximated by a polynomial in z.*

In the special case that the region C is convex with respect to the origin, the proof is elementary. The function $f[nz/(n+1)]$ is defined and analytic throughout the region $C^{(n)}$ found from C by stretching the plane away from the origin in the ratio n to $n + 1$, by the transformation $z' = (n + 1)z/n$. Let $\epsilon > 0$ be given. The function $f[nz/(n + 1)]$ is defined at all points of $\bar{C}$, and at all points z of $\bar{C}$ we have by the uniform continuity of $f(z)$ in $\bar{C}$, for a suitably chosen n,

$$|f(z) - f[nz/(n + 1)]| < \epsilon/2, \qquad z \text{ in } \bar{C}.$$

There exists (§1.6, Theorem 8) a polynomial $p(z)$ such that we have

$$|f[nz/(n + 1)] - p(z)| < \epsilon/2, \qquad z \text{ in } \bar{C},$$

for the function $f[nz/(n + 1)]$ is analytic in $\bar{C}$. Then we have

$$|f(z) - p(z)| < \epsilon, \qquad z \text{ in } \bar{C},$$

and the proof is complete for this special case.

The proof of Theorem 5 in the general case follows the proof just given, except that now we apply Corollary 1 to Theorem 4 instead of using the simple linear transformation. Let the present region C play the rôle of the region T in Theorem 4, and let the T_n have the properties there required. If C is mapped onto the interior of T_n, the function $f(z)$ is transformed into a function $f[\chi_n(z)]$ (notation of Corollary 1 to Theorem 4) which is analytic in $\bar{C}$. If $\epsilon > 0$ is given, there exists n such that

$$|f(z) - f[\chi_n(z)]| < \epsilon/2$$

for z in $\bar{C}$. There exists (§1.6, Theorem 8) a polynomial $p(z)$ such that we have

$$|f[\chi_n(z)] - p(z)| < \epsilon/2$$

for z in $\bar{C}$. Thus we have

$$|f(z) - p(z)| < \epsilon, \qquad z \text{ in } \bar{C},$$

and Theorem 5 is completely proved.

In particular, it will be noticed that the mapping function $w = \Phi(z)$ satisfies the requirements of this theorem, and hence can be uniformly approximated by a polynomial in z. Otherwise expressed, the conformal map of C onto γ can be performed approximately in $\bar{C}$ by a polynomial in z.

The following corollary is needed in §2.5, but is to be broadly generalized in §11.1:

COROLLARY. *Under the conditions of Theorem 5, let the point $z = \alpha$ lie in $\bar{C}$. Then the function $f(z)$ can be approximated in $\bar{C}$ by a polynomial which takes on the value $f(\alpha)$ in the point $z = \alpha$.*

Let $\epsilon > 0$ be arbitrary. There exists a polynomial $p(z)$ such that we have

$$|f(z) - p(z)| < \epsilon/2, \qquad z \text{ in } \bar{C};$$

in particular we have

$$|f(\alpha) - p(\alpha)| < \epsilon/2.$$

We may now write

$$|f(z) - [p(z) - p(\alpha) + f(\alpha)]| < \epsilon, \qquad z \text{ in } \bar{C},$$

so the polynomial $p(z) - p(\alpha) + f(\alpha)$ fulfills the required conditions.

Theorem 5 extends to the case of a multiply connected region:

THEOREM 6. *Let C be a region of the extended z-plane bounded by a finite number of Jordan curves $J_1, J_2, \cdots, J_\nu$, no two of which have a common point. Let the function $f(z)$ be analytic in C, continuous in $\bar{C}$. Then in $\bar{C}$ the function $f(z)$ can be uniformly approximated by a rational function of z. If points $z_1, z_2, \cdots, z_\nu$ are chosen, separated respectively from the interior of C by the curves $J_1, J_2, \cdots, J_\nu$, then this rational function can be chosen to have all its poles in these points z_k.*

It is sufficient for us to consider the latter part of the theorem. Let C_k denote that Jordan region bounded by J_k which does not contain z_k; then C lies in C_k. By §1.7, the function $f(z)$ can be expressed in $\bar{C}$ as the sum of ν functions $f_k(z)$ analytic respectively interior to the regions C_k, continuous in the closed regions $\bar{C}_k$. By Theorem 5, the function $f_k(z)$ can be uniformly approximated in $\bar{C}_k$ by a rational function whose only pole lies in the point z_k. Thus $f(z)$ can be uniformly approximated in $\bar{C}$ by a rational function whose only poles lie in the points z_k. The proof is complete.

From the discussion just given we can state also the

COROLLARY. *Under the hypothesis of Theorem 6, the function $f(z)$ can be expressed in $\bar{C}$ as the sum of ν functions $f_k(z)$ all analytic in C and analytic respectively in a Jordan region C_k containing C bounded by J_k. In the region $\bar{C}_k$ the function $f_k(z)$ can be expressed as the limit of a uniformly convergent sequence of polynomials in $1/(z - z_k)$ [or in z if $z_k = \infty$].*

Theorems 5 and 6 admit application to the study of approximation of functions merely known to be continuous without reference to analyticity.

THEOREM 7. *Let C be a Jordan curve of the finite z-plane containing in its interior the origin. Then an arbitrary function $f(z)$ continuous on C can be uniformly approximated on C by a polynomial in z and $1/z$.*

Weierstrass's theorem on approximation by trigonometric polynomials asserts that *a function $f(\theta)$ which is continuous for all θ and of periodicity 2π can be uniformly approximated for all values of θ as closely as desired by a trigonometric polynomial in θ of the form*

$$\sum_{n=0}^{N} (a_n \cos n\theta + b_n \sin n\theta) .$$

Theorem 7 is a generalization of Weierstrass's theorem, for on the unit circle $\gamma: |z| = 1$ we have in polar coordinates

$$z^n = \cos n\theta + i \sin n\theta , \qquad z^{-n} = \cos n\theta - i \sin n\theta ,$$

$$\sin n\theta = \frac{z^n - z^{-n}}{2i}, \qquad \cos n\theta = \frac{z^n + z^{-n}}{2}.$$

On γ a polynomial in z and $1/z$ of degree N is a trigonometric polynomial in θ of order N, and conversely; a function continuous on γ is a function of θ continuous for all θ and of period 2π, and conversely. To be sure, Weierstrass's theorem is ordinarily stated merely for the case of a real function, but that theorem for a real function implies directly the theorem for a complex function, and reciprocally.

Let the interior of C be mapped onto the interior of $\gamma: |w| = 1$ by the transformation $w = \Phi(z)$, $z = \Psi(w)$. Let an arbitrary positive ϵ be given. By Weierstrass's theorem there exists a polynomial $F(w, 1/w)$ in w and $1/w$ such that we have

$$|f[\Psi(w)] - F(w, 1/w)| < \epsilon/2, \qquad w \text{ on } \gamma ,$$

that is,

$$|f(z) - F[\Phi(z), 1/\Phi(z)]| < \epsilon/2, \qquad z \text{ on } C .$$

The function $F[\Phi(z), 1/\Phi(z)]$ is analytic in an annular region R bounded by C and by an arbitrary circle interior to C whose center is the origin, and is continuous in $\bar{R}$. By Theorem 6, there exists a polynomial $P(z, 1/z)$ in z and $1/z$ such that we have

$$|F[\Phi(z), 1/\Phi(z)] - P(z, 1/z)| < \epsilon/2, \qquad z \text{ in } \bar{R}.$$

This inequality holds in particular for z on C, so we have

$$|f(z) - P(z, 1/z)| < \epsilon, \qquad z \text{ on } C ;$$

the proof is complete.

Theorem 7 is to be used in the proof of

THEOREM 8. *Let C be a Jordan arc of the finite z-plane. Then an arbitrary function $f(z)$ continuous on C can be uniformly approximated on C by a polynomial in z.*

Theorem 8 is a generalization of Weierstrass's theorem on approximation by polynomials, which is the special case of Theorem 8 in which C is a segment of the axis of reals. Of course, Weierstrass's theorem is ordinarily stated for the approximation of a real function, but the theorem for real functions gives an immediate proof of the theorem for complex functions, and conversely.

In the proof of Theorem 8 it is no loss of generality to choose the origin of coordinates to lie off the Jordan arc C. Let C' be a Jordan curve containing the origin in its interior and of which C is an arc; such a curve exists. Let the definition of the function $f(z)$ be extended so that $f(z)$ is defined and continuous at every point of C'; it is sufficient, for instance, to define $f(z)$ at the points of C' not belonging to C by the formula

$$w_1 + (w_2 - w_1)(z - z_1)/(z_2 - z_1),$$

where w_1 and w_2 are the respective values of $f(z)$ in the end-points z_1 and z_2 of C.

Let an arbitrary positive ϵ be given. By Theorem 7 there exists a polynomial $P(z, 1/z)$ in z and $1/z$ such that we have

$$|f(z) - P(z, 1/z)| < \epsilon/2, \qquad z \text{ on } C';$$

in particular this inequality is valid for z on C. Let R be a bounded Jordan region which contains C in its interior, but to which the origin is exterior. The function $P(z, 1/z)$ is analytic in $\bar{R}$, so there exists a polynomial $p(z)$ in z such that

$$|P(z, 1/z) - p(z)| < \epsilon/2, \qquad z \text{ in } \bar{R};$$

this inequality is of course valid for z on C. Then for z on C we have

$$|f(z) - p(z)| < \epsilon,$$

and the proof is complete.

Theorems 5–8 are due to Walsh [1926, 1926a]; the special case of Theorem 5 that C is convex had been treated previously and independently by Hilb and Szász [1924] and Walsh [1924]. Theorems 5 and 7 for the case of analytic Jordan curves were proved by Walsh [1924].

§2.5. Applications, Jordan configurations

Theorem 7 admits a number of applications other than to Theorem 8.

THEOREM 9. *Let C be an arbitrary Jordan curve of the finite z-plane. Then any function $f(z)$ continuous on C can be uniformly approximated on C by the sum of a polynomial in z and a polynomial in $\bar{z}$.*

Let the functions $w = \Phi(z)$, $z = \Psi(w)$, map the interior of C onto the interior of γ: $|w| = 1$. The function $f[\Psi(w)]$ can be uniformly approximated on γ by the

sum of a polynomial in w and a polynomial in $1/w$, hence by the sum of a polynomial in w and a polynomial in $\bar{w}$. A polynomial in w corresponds to a function analytic interior to C, continuous in the corresponding closed region, and can therefore be uniformly approximated on C by a polynomial in z. By taking conjugates, a polynomial in $\bar{w}$ can be uniformly approximated on C by a polynomial in $\bar{z}$, so the theorem follows.

A related theorem [Walsh, 1928; compare also §6.10, Theorem 14] is

THEOREM 10. *Let C be an analytic Jordan curve of the finite z-plane or more generally a Jordan curve whose interior can be mapped conformally onto the interior of γ: $|w| = 1$ by a transformation $w = \Phi(z)$, $z = \Psi(w)$, so that $\Phi'(z)$ exists and is continuous throughout the closed interior $\bar{C}$ of C. Let $f(z)$ be continuous on C. Then the following conditions are all equivalent:*

a) $$\int_C f(z)\, z^n\, dz = 0, \qquad n = 0, 1, 2, \cdots;$$

b) $$\int_C \frac{f(t)\, dt}{t - z} \equiv 0, \qquad z \text{ exterior to } C;$$

c) $$\int_C f(z)\, \omega(z)\, dz = 0,$$

for every function $\omega(z)$ analytic on and within C;

d) $$\int_C f(z)\, \omega(z)\, dz = 0,$$

for every function $\omega(z)$ analytic interior to C, continuous in $\bar{C}$;

e) *there exists a function analytic interior to C, continuous in $\bar{C}$, which coincides with $f(z)$ on C.*

Condition b) seems more analogous to the conditions c) and d) if the notation is slightly changed:

b′) $$\int_C \frac{f(z)\, dz}{z - \zeta} \equiv 0, \qquad \zeta \text{ exterior to } C;$$

it is to be noticed that the function $1/(z - \zeta)$ is analytic on and interior to C, and hence b′) is a condition of the same form as c) and d). The original notation of b) is used because that is the ordinary notation for the representation of a function by Cauchy's integral; condition b) expresses the identical vanishing of a function defined by an integral of Cauchy type.

It is obvious, by Cauchy's integral theorem (see also Theorem 11 below), that e) implies d), and we see at once that d) implies c) and c) implies a) and b′). It remains merely to show that a) and b) imply e).

Any function $\omega(z)$ analytic on and within C can be expressed in $\bar{C}$ as a uniformly convergent sequence of rational functions (by §1.5, Theorem 7 and §1.4, Theorem 5) of the form:

$$\omega(z) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{A_{nk}}{z - \alpha_{nk}}, \tag{12}$$

where the poles α_{nk} lie exterior to C. Condition b′) now implies c) and hence a), if the integral in c) is computed by means of the substitution (12) and term-by-term integration.

We now show that a) implies e). The case that C is the unit circle can be treated easily. The formal Laurent development of $f(z)$ on C is of the form of a Taylor series, for by a) the coefficients of the negative powers of z vanish:

$$f(z) \sim a_0 + a_1 z + a_2 z^2 + \cdots, \qquad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{z^{n+1}}.$$

This development is precisely the formal Fourier development of $f(z)$ on the interval $0 \leqq \theta \leqq 2\pi$ if we set $z = e^{i\theta}$. The Fourier development, when summed by the method of Cesàro, converges uniformly to $f(z)$ on C, by the continuity of $f(z)$ on C (Fejér). Each term of the corresponding sequence is analytic on and interior to C, hence the limit of the sequence is analytic interior to C, continuous in the corresponding closed region, and equals $f(z)$ on C. This completes the proof when C is the unit circle.

If C is not the unit circle, we study the function $[\Phi(z)]^n \Phi'(z)$, where n is a non-negative integer. This function is analytic within C, continuous in $\bar{C}$, hence can be expressed

$$[\Phi(z)]^n \, \Phi'(z) = \sum_{k=1}^{\infty} p_k(z), \qquad z \text{ in } \bar{C},$$

where the $p_n(z)$ are polynomials in z, and convergence is uniform in $\bar{C}$. If we make use of a), term-by-term integration yields

$$\int_C f(z)\, [\Phi(z)]^n \, \Phi'(z)\, dz = \int_\gamma f[\Psi(w)] w^n \, dw = 0, \qquad n = 0, 1, 2, \cdots.$$

By the special case already proved, there exists a function analytic interior to γ, continuous in the corresponding closed region, equal to $f[\Psi(w)]$ on γ; this is equivalent to e).

We leave to the reader the analogous proof of the

COROLLARY. *Let C be a finite analytic Jordan curve of the extended z-plane or more generally a Jordan curve whose exterior can be mapped conformally onto the exterior of γ: $|w| = 1$ by a transformation $w = \phi(z)$ so that $\phi'(z)$ exists and is continuous throughout the closed exterior of C. Let the origin $z = 0$ lie interior to C. Let $f(z)$ be continuous on C. Then the following conditions are all equivalent:*

a) $$\int_C f(z)\, z^n \, dz = 0, \qquad n = -1, -2, -3, \cdots;$$

b) $$\int_C \frac{f(t)\, dt}{t - z} \equiv 0, \qquad z \text{ interior to } C;$$

c) $$\int_C f(z)\, \sigma(z)\, dz = 0,$$

for every function $\sigma(z)$ analytic on and exterior to C and zero at infinity;

d) $$\int_C f(z)\,\sigma(z)\,dz = 0,$$

for every function $\sigma(z)$ analytic exterior to C, continuous in the corresponding closed region, and zero at infinity;

e) *there exists a function analytic exterior to C, continuous in the corresponding closed region, zero at infinity, and coinciding with $f(z)$ on C.*

In the proof, it is convenient to use the Corollary to Theorem 5.

Theorem 10 also admits of generalizations to multiply connected regions; compare Walsh [1928].

§2.6. General forms of Cauchy's integral formula

In the sequel, we shall frequently need to use Cauchy's integral formula in rather general situations, and it happens that the methods of approximation of the present chapter yield useful theorems. This method (although not the theorems) is due to Walsh [1933a].

THEOREM 11. *Let R be a limited region bounded by rectifiable Jordan curves $C_1, C_2, \cdots, C_\nu$, of which no two have a point in common. Let the function $F(z)$ be analytic interior to R, continuous in the corresponding closed region $\bar{R}$. Then we have*

$$\int_B F(z)\,dz = 0, \tag{13}$$

where B is the boundary of R.

Let us first prove (13) for the case that $F(z)$ is a rational function. The integral

$$\int_{C_j} F(z)\,dz \tag{14}$$

is defined as the limit of the sum

$$\sum_{k=1}^{n} F(z_k)\,\Delta_k z, \qquad \Delta_k z = z_{k+1} - z_k,$$

when the points z_k depend on n and are chosen on C_j so that $\max|\Delta_k z|$ approaches zero with $1/n$. By the uniform continuity of $F(z)$ in a region containing C_j in its interior, and by the rectifiability of C_j, it follows that (14) is also the limit of the integral

$$\int_{\Gamma_j^{(n)}} F(z)\,dz, \tag{15}$$

where $\Gamma_j^{(n)}$ is the closed polygon $z_1 z_2 \cdots z_n z_1$:

$$\left| \int_{\Gamma_j^{(n)}} F(z)\, dz - \sum_{k=1}^{n} F(z_k)\, \Delta_k z \right| \leqq l_n \max\, [\,| F(z) - F(z_k) |\,,\ z \text{ on } \overline{z_k z_{k+1}}\,],$$

where l_n is the length of $\Gamma_j^{(n)}$; as n becomes infinite l_n approaches the length of C_j. The sum of the integrals (15) for $j = 1, 2, \cdots, \nu$ is zero if the quantities $\Delta_k z$ are chosen so small that no poles of $F(z)$ lie on the $\Gamma_j^{(n)}$ or within the region bounded by all the $\Gamma_j^{(n)}$. Hence the sum of the integrals (14) for $j = 1, 2, \cdots, \nu$ is also zero, and the theorem is proved when $F(z)$ is rational.

In the more general case, the function $F(z)$ can be expressed in $\bar{R}$ as the sum of a uniformly convergent series of rational functions whose poles lie exterior to $\bar{R}$. For each of these rational functions the equation corresponding to (13) is valid. The integral in (13) for $F(z)$ is the limit of the corresponding integrals for the rational functions, and hence is zero. The proof is complete.

A consequence of Theorem 11 is

THEOREM 12. *Let R be a limited region bounded by a finite number of rectifiable Jordan curves of which no two have a point in common. Let the function $f(z)$ be analytic interior to R, continuous in the corresponding closed region $\bar{R}$. Then we have*

$$f(z) = \frac{1}{2\pi i} \int_B \frac{f(t)\, dt}{t - z}, \qquad z \text{ interior to } R, \tag{16}$$

where B is the boundary of R.

Let C denote a circle whose center is z and whose radius is less than the distance from z to B. We have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{t - z}. \tag{17}$$

But the function $f(t)/(t - z)$ is a function of t analytic in the region R' common to R and the exterior of C, continuous in the corresponding closed region $\bar{R}'$. Theorem 11 yields the equation

$$\int_{B'} \frac{f(t)\, dt}{t - z} = 0, \tag{18}$$

where B' is the boundary of R'. Equation (18) together with (17) now implies (16), if the various senses of integration are taken into account.

It is hardly necessary to add that Chapter I and §§2.1–2.4 do not require general forms of Cauchy's integral theorem or formula, so that our application of Theorem 6 is valid here. For the proof of Theorem 6 it is sufficient to prove Cauchy's integral theorem and formula for regions bounded by *contours*, and for functions analytic in the corresponding closed regions.

Theorem 12 extends in the usual way to regions containing the point at infinity in their interiors. Another extension is somewhat less obvious; we merely sketch the proof:

THEOREM 13. *Let C be a rectifiable Jordan arc of the z-plane. Let the function $f(z)$ be analytic in the extended plane exterior to C, zero at infinity, continuous in the plane cut along C. Then we have*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{t - z}, \qquad z \text{ exterior to } C, \tag{19}$$

where the integral is taken over the boundary of the cut plane in the positive sense.

Denote by α and β the end-points of C, and draw circles γ_1 and γ_2 with centers α and β respectively and radius $\delta < |\alpha - \beta|/2$. Denote by C_δ the curve composed of the two circles γ_1 and γ_2, together with the two banks of the arc of C whose initial point is the last point from α of intersection of C with γ_1 and whose terminal point is the first point from α of intersection of C with γ_2. Transform the z-plane (to which C belongs) by setting

$$\frac{w - \alpha}{w - \beta} = \left(\frac{z - \alpha}{z - \beta}\right)^{n/(n+1)}, \tag{20}$$

where n is an integer that will be allowed to become infinite. This transformation leaves α and β invariant, and if we fasten our attention on a suitable sheet of the Riemann surface for w, the exterior of C_δ is transformed into the exterior of a Jordan curve $C_\delta^{(n)}$, with the point at infinity invariant. The curve $C_\delta^{(n)}$ is rectifiable, and we have from Theorem 12

$$f[z(w)] = \frac{1}{2\pi i} \int_{C_\delta^{(n)}} \frac{f[z(\tau)]\, d\tau}{\tau - w}, \qquad w \text{ exterior to } C_\delta^{(n)}, \tag{21}$$

where $z(w)$ is the appropriate branch of the function defined by (20). When n becomes infinite, the points w approach the corresponding points z uniformly for all values of z, and the function $f[z(w)]$ approaches $f(z)$ uniformly. The derivative dw/dz approaches unity uniformly at every point of C_δ, so we have from (21)

$$f(z) = \frac{1}{2\pi i} \int_{C_\delta} \frac{f(t)\, dt}{t - z}, \qquad z \text{ exterior to } C_\delta. \tag{22}$$

When δ approaches zero the right-hand member of (22) remains unchanged in value. But the function $f(z)$ is uniformly limited for all values of z, so the part of the integral in (22) which is taken over γ_1 and γ_2 approaches zero, and the integral over the remainder of C_δ approaches the integral over C. A given point z exterior to C is also exterior to C_δ for sufficiently small δ, so (19) is established.

Iteration of the method of Theorem 13 yields still more general results.

§2.7. Surface integrals as measures of approximation

We have hitherto considered maximum error as the measure of approximation of one function to another. There are, however, other important measures of approximation, one of which is now to be treated in

THEOREM 14. *Let C be a limited simply connected region of the z-plane whose boundary is also the boundary of an infinite region. Let the function $f(z)$ be analytic interior to C and such that*

$$\iint_C |f(z)|^p \, dS, \qquad p > 0,$$

exists. Let $\epsilon > 0$ be given. Then there exists a polynomial $P(z)$ such that we have

$$\iint_C |f(z) - P(z)|^p \, dS < \epsilon. \tag{23}$$

If the boundary of C possesses area, these and later integrals are to be taken over the *interior* of C.

Let $T_1, T_2, \cdots$ be a sequence of Jordan regions whose boundaries lie exterior to C, which contain C in their interiors, and which approach C monotonically in the sense of Corollary 2 to Theorem 4. In the notation of §2.3, let us set

$$f_n(z) = f[\chi_n(z)]\,[\chi_n'(z)]^{2/p},$$

which is defined and analytic throughout the interior of T_n if a particular determination of the fractional power is chosen. Let C' denote a closed Jordan region interior to C, to be determined in more detail later. We may write

$$\iint_C |f(z) - f_n(z)|^p \, dS = \iint_{C'} |f(z) - f_n(z)|^p \, dS + \iint_{C-C'} |f(z) - f_n(z)|^p \, dS, \tag{24}$$

and it follows from Theorem 4, Corollary 2 that equation (11) is valid uniformly on C', together with the equation

$$\lim_{n\to\infty} \chi_n'(z) = 1.$$

Then the limit of $f_n(z)$ is $f(z)$ uniformly on C', provided the appropriate fractional power is used in the definition of $f_n(z)$. Hence the first integral in the right-hand member of (24) approaches zero with $1/n$.

The second integral in the right-hand member of (24) can be treated by writing

$$\iint_{C-C'} |f(z) - f_n(z)|^p \, dS \leqq A \iint_{C-C'} |f(z)|^p \, dS + A \iint_{C-C'} |f_n(z)|^p \, dS, \tag{25}$$

where A is a constant depending on p but not on n. Inequality (25) is a consequence of §5.2, inequalities (10), which are to be used frequently in the sequel. The second integral in the right-hand member of (25) can be transformed by writing

$$\iint_{C-C'} |f_n(z)|^p \, dS = \iint_{D_n} |f(z')|^p \, dS', \qquad dS' = |\chi_n'(z)|^2 \, dS, \tag{26}$$

where D_n is the image of $C - C'$ under the transformation $z' = \chi_n(z)$.

Let $\delta > 0$ be given. Choose C'', a closed set interior to C, so that

$$A \iint_{C-C''} |f(z)|^p \, dS < \delta/3 ; \tag{27}$$

whenever C' contains C'' in its interior, we also have

$$A \iint_{C-C'} |f(z)|^p \, dS < \delta/3 . \tag{28}$$

Let C' be now arbitrary, containing C'' in its interior. For sufficiently large n the point set D_n lies in $C - C''$, by §2.3, Corollary 1 to Theorem 4, so we have by (26) and (27)

$$A \iint_{C-C'} |f_n(z)|^p \, dS < \delta/3 . \tag{29}$$

We have already indicated that for n sufficiently large the first integral in the right-hand member of (24) is less than $\delta/3$, so we have by (28), (29), and (25) for sufficiently large n

$$\iint_C |f(z) - f_n(z)|^p \, dS < \delta . \tag{30}$$

The function $f_n(z)$ is analytic throughout the interior of T_n, hence is analytic not merely in the closed region $\bar{C}$, but also in every point of the plane separated by $\bar{C}$ from the point at infinity. Hence $f_n(z)$ can be uniformly approximated on $\bar{C}$ by a polynomial $P(z)$, therefore approximated in the sense

$$\iint_C |f_n(z) - P(z)|^p \, dS < \delta_1 , \tag{31}$$

where $\delta_1 > 0$ is preassigned. Inequality (23) is now a consequence of (30) and (31), again by virtue of §5.2, inequalities (10).

Theorem 14 is due to Carleman [1922] in the case that C is a Jordan region; the proof given by Farrell [1934] is valid in the more general case. Carleman omits the details, simply stating that the theorem follows from results due to Lindelöf; this seems to be the first application of the modern theory of conformal mapping to questions of possibility of approximation.

§2.8. Uniform approximation; further results

Certain further results on uniform approximation lie immediately at hand. For instance, suppose a function $f(z)$ can be uniformly approximated by a polynomial on each of two point sets Γ_1 and Γ_2, contained respectively in mutually exterior closed finite Jordan regions C_1 and C_2. Then $f(z)$ can be uniformly approximated by a polynomial simultaneously on Γ_1 and Γ_2. Indeed, let $\epsilon > 0$ be given. There exist polynomials $p_1(z)$ and $p_2(z)$ such that

$$|f(z) - p_1(z)| < \epsilon/2, \quad z \text{ on } \Gamma_1 ; \qquad |f(z) - p_2(z)| < \epsilon/2, \quad z \text{ on } \Gamma_2 .$$

There exists also a polynomial $p(z)$ such that

$$|p_1(z) - p(z)| < \epsilon/2, \quad z \text{ on } C_1; \qquad |p_2(z) - p(z)| < \epsilon/2, \quad z \text{ on } C_2,$$

and this implies

$$|f(z) - p(z)| < \epsilon, \quad z \text{ on } \Gamma_1; \qquad |f(z) - p(z)| < \epsilon, \quad z \text{ on } \Gamma_2,$$

as we were to prove. This remark obviously extends to any finite number of point sets $\Gamma_1, \Gamma_2, \cdots, \Gamma_\nu$.

Extension of this reasoning, together with the methods of conformal mapping as used in the present chapter, yields various additional results on possibility of uniform (and other) approximation. For the sake of economy of space, we content ourselves with the statement of a single example [Walsh, 1928a, 1929b]:

THEOREM 15. *Let C be a closed point set of the extended plane, bounded by a finite number of Jordan arcs or Jordan curves or both, no two of which have more than a finite number of points in common. Let the function $f(z)$ be analytic in the interior points of C, continuous on the closed point set. Let points $z_1, z_2, \cdots$ be assigned, not necessarily infinite in number, at least one in each of the regions into which C separates the plane. Then the function $f(z)$ can be uniformly approximated on C by a rational function whose poles lie in the points z_k.*

In particular, if C is limited and is the complement of an infinite region, then $f(z)$ can be uniformly approximated on C by a polynomial.

Chapter I, Theorem 6 and Chapter II, Theorems 5–8 are special cases of Theorem 15.

It is clear that the discussion of §1.10 enables us to state a converse of Theorem 15; with the help of Theorem 15 itself we formulate

THEOREM 16. *Let C be a closed point set of the extended plane, bounded by a finite number of Jordan arcs or Jordan curves or both, no two of which have more than a finite number of points in common. Let points $z_1, z_2, \cdots$ not necessarily infinite in number, be given exterior to C. A necessary and sufficient condition that EVERY function $f(z)$ analytic in the interior points of C and continuous on the closed point set can be uniformly approximated on C by a rational function whose poles lie in the points z_k, is that at least one point z_k should lie in each of the regions into which C separates the plane. A necessary and sufficient condition that every function $f(z)$ continuous on C can be uniformly approximated on C by a rational function whose poles lie in the points z_k, is that C should have no interior points, and that at least one point z_k should lie in each of the regions into which C separates the plane.*

If C has interior points, then any function $f(z)$ which can be uniformly approximated on C by rational functions whose poles lie in the z_k must be analytic in the interior points of C, for a sequence of such rational functions converging uniformly on C represents a function analytic in every interior point of C.

If C is a closed limited point set, and if an arbitrary function $f(z)$ continuous on C can be uniformly approximated on C by a polynomial in z, it is clear from the

proof of Theorem 16 that C can have no interior points and cannot separate the plane. Quite recently, Lavrentieff [1934] has established the important converse of this proposition, a generalization of Theorem 8: *Let C be an arbitrary closed limited point set without interior points which does not separate the plane. Then an arbitrary function $f(z)$ continuous on C can be uniformly approximated on C by a polynomial in z.*

The corresponding problem for approximation by rational functions with assigned poles is still unsolved, as is the problem of determining the most general closed point set C such that an arbitrary function analytic in the interior points of C and continuous on C can be uniformly approximated on C by polynomials or more generally by rational functions with preassigned poles.

CHAPTER III

INTERPOLATION AND LEMNISCATES

§3.1. Polynomials of interpolation

One of the most important methods of effectively determining polynomials approximating to a given function is by the use of interpolation. In the present chapter we shall study the more elementary facts regarding interpolation by polynomials, and in the present chapter we deal entirely with the finite plane.

THEOREM 1. *Let the distinct points* $z_1, z_2, \cdots, z_{n+1}$ *and values* $w_1, w_2, \cdots, w_{n+1}$ *be given. Then there exists a unique polynomial* $p(z)$ *of degree* n *which takes on the values* w_k *in the points* z_k.

The existence of the polynomial

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

depends on the solution of a system of $n + 1$ equations for the $n + 1$ unknowns a_k:

$$\begin{aligned} a_0 z_1^n + a_1 z_1^{n-1} + \cdots + a_n &= w_1 , \\ a_0 z_2^n + a_1 z_2^{n-1} + \cdots + a_n &= w_2 , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &, \\ a_0 z_{n+1}^n + a_1 z_{n+1}^{n-1} + \cdots + a_n &= w_{n+1} . \end{aligned} \tag{1}$$

The determinant Δ here is a function of the z_k not difficult to evaluate, but the exact evaluation is immaterial for our present purpose. The vanishing of Δ is a necessary and sufficient condition that the homogeneous system corresponding to (1) (i.e., with the right-hand members replaced by zeros) should admit a solution $a_0, a_1, \cdots, a_n$ of numbers not all zero, or in other words is a necessary and sufficient condition that there should exist a polynomial of degree n whose coefficients a_k are not all zero which vanishes in the $n + 1$ distinct points z_j. The latter eventuality cannot occur, so Δ is different from zero, the system (1) uniquely determines the coefficients a_k, and the theorem is established.

The proof just given carries over without change to yield the following more general result:

THEOREM 2. *Let the distinct points* $z_1, z_2, \cdots, z_k$ *and values* $w_j^{(0)}, w_j^{(1)}, \cdots, w_j^{(m_j)}$, $j = 1, 2, \cdots, k$, *be given. Then there exists a unique polynomial* $p(z)$ *of degree* $n = -1 + \sum_{j=1}^{k} (m_j + 1)$ *which satisfies the conditions*

$$p^{(\nu)}(z_j) = w_j^{(\nu)}, \qquad \nu = 0, 1, \cdots, m_j; \qquad j = 1, 2, \cdots, k . \tag{2}$$

Here (and in the sequel) the notation $p^{(0)}(z_j)$ indicates $p(z_j)$, and $p^{(\nu)}(z_j)$, $\nu > 0$, indicates the ν-th derivative of $p(z)$ at the point $z = z_j$. We frequently describe the situation of Theorem 2 by saying that the points z_j are given, *of respective multiplicities* $m_j + 1$.

In particular, the values $w_j^{(\nu)}$ may be given by means of a function $f(z)$:

$$w_j^{(\nu)} = f^{(\nu)}(z_j), \qquad \nu = 0, 1, \cdots, m_j; \qquad j = 1, 2, \cdots, k.$$

In this case we say that $p(z)$ *coincides with or interpolates to* the function $f(z)$ in the points z_j, considered of respective multiplicities $m_j + 1$. Another way of describing this same fact is to say that the function $f(z) - p(z)$ vanishes in the points z_j, considered of respective multiplicities $m_j + 1$; the multiplicities of the respective zeros of the function $f(z) - p(z)$ may of course be greater than $m_j + 1$, and indeed this function may vanish identically.

There are various formulas for interpolating polynomials. Let us return to the situation of Theorem 1. The polynomial $p(z)$ is known to be unique and is clearly the sum of $n + 1$ unique polynomials of degree n each of which takes on the value w_k in the point z_k and vanishes in the remaining points z_j. These latter polynomials are not difficult to write down in terms of the polynomial of degree n which takes on the value unity in the point z_k and vanishes in the points z_j, $j \neq k$. Thus we have *Lagrange's interpolation formula*:

$$(3) \quad p(z) = \sum_{k=1}^{n+1} w_k \frac{\omega(z)}{(z - z_k)\,\omega'(z_k)}, \qquad \omega(z) = (z - z_1)(z - z_2) \cdots (z - z_{n+1}).$$

Of course the fraction in (3) is not defined for $z = z_k$, but (here and in all similar cases in the sequel) we suppose the function to be defined for the exceptional value by allowing the variable to approach that value. Verification of (3) is then immediate, for we have

$$\lim_{z \to z_k} \left[\frac{\omega(z)}{z - z_k}\right] = \lim_{z \to z_k} \left[\frac{\omega(z) - \omega(z_k)}{z - z_k}\right] = \omega'(z_k);$$

the polynomial $\omega(z)/(z - z_k)$ is of degree n. An analogous formula exists for the more general situation of Theorem 2, the study of which is left to the reader.

There is a particularly useful formula, due to Hermite, for the polynomial $p(z)$ of degree n which interpolates to an analytic function $f(z)$ in the $n + 1$ points $z_1, z_2, \cdots, z_{n+1}$ not necessarily distinct:

$$(4) \qquad f(z) - p(z) = \frac{1}{2\pi i}\int_C \frac{\omega(z)\, f(t)\, dt}{\omega(t)\,(t - z)}, \qquad z \text{ interior to } C,$$

$$(5) \qquad p(z) = \frac{1}{2\pi i}\int_C \frac{\omega(t) - \omega(z)}{\omega(t)\,(t - z)} f(t)\, dt,$$

where $\omega(z)$ is defined as in (3), and where $f(z)$ is analytic on and within the contour C containing the points z_k in its interior or more generally is analytic on and within each of a set C of mutually exterior contours containing the points z_k

in their interiors. Equations (4) and (5) are clearly equivalent for z interior to C, by virtue of Cauchy's integral formula. The proof of their validity is immediate; from equation (4) we see that $f(z)$ and $p(z)$ coincide in the $n + 1$ points z_k, distinct or not, for the function

$$\frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{\omega(t)(t - z)}$$

is analytic interior to C, hence analytic in each of the points z_k. In equation (5) we see that the integrand has no singularity in z, for the numerator vanishes identically for $t = z$ and hence is divisible by $t - z$. The present numerator in (5) is a polynomial in z of degree $n + 1$, so the function $p(z)$ defined by (5) is a polynomial in z of degree n; the proof of (4) is complete. Moreover, since equation (5) is valid for z interior to C, that equation is also valid for all values of z; we naturally suppose the integrand defined for $z = t$ by a limiting process as z approaches t.

It is sufficient here in the derivation of (4) and (5) if $f(z)$ is analytic interior to C and continuous on and within the contours or mutually exterior rectifiable Jordan curves C. More generally—and this remark will find application later—it is sufficient if the function $f(z)$ is integrable on C and if Cauchy's integral formula is valid:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t - z}, \qquad z \text{ interior to } C,$$

irrespective of any interpretation of $f(z)$ on C as boundary values of the function $f(z)$ interior to C.

Formulas (4) and (5) can be motivated as follows. Choose the particular case

$$f(z) \equiv \frac{1}{t - z}, \qquad t \neq z_k.$$

The formula

(6) $$p(z) \equiv \frac{1}{t - z} - \frac{\omega(z)}{\omega(t)(t - z)}$$

can be verified precisely as we have verified (4) and (5), and can in fact be derived without great difficulty by a study of the system (1). The polynomial $p(z)$ in (6) is a polynomial in z of degree n (whose coefficients are functions of t), and if that polynomial is multiplied by $f(t)\,dt/(2\pi i)$ and integrated with respect to t over an arbitrary closed contour or set of mutually exterior contours C, where $f(z)$ is an arbitrary function analytic on and within C, the result is still a polynomial in z of degree n. If C is chosen to contain the points z_k in its interior, and if the new polynomial is denoted by $P(z)$, we have from (6) an equation equivalent to (4) and (5):

$$P(z) \equiv f(z) - \frac{1}{2\pi i}\int_C \frac{\omega(z)\,f(t)\,dt}{\omega(t)(t - z)}, \qquad z \text{ interior to } C,$$

and it is clear that $P(z)$ interpolates to $f(z)$ in the points z_k.

In the study of expansion of analytic functions, it is often of great convenience to study first the expansion of the function $1/(t - z)$. That fact appears by inspection of the reasoning used in §§1.4 and 1.6, it will appear in much of the later material in the present work, and it indicates the naturalness of comparing (4) and (5) with (6).

§3.2. Sequences and series of interpolation

The general problem of the representation of an analytic function by a sequence of polynomials found by interpolation is the following. Let the points (not necessarily distinct)

$$
\begin{array}{ll}
 & z_1^{(0)}, \\
 & z_1^{(1)}, z_2^{(1)}, \\
(7) & \cdots\cdots\cdots, \\
 & z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}, \\
 & \cdots\cdots\cdots\cdots\cdots\cdots,
\end{array}
$$

be given and let the function $f(z)$ be analytic in the points $z_k^{(n)}$. Let $p_n(z)$ be the polynomial of degree n found by interpolation to $f(z)$ in the points $z_k^{(n)}$, $k = 1, 2, \cdots, n + 1$. The problem is *to study the convergence of the sequence* $p_n(z)$ *to the function* $f(z)$.

If the points (7) are chosen interior to a region R within which $f(z)$ is analytic, it does not necessarily follow that we have

$$\lim_{n \to \infty} p_n(z) = f(z)$$

for every z interior to R. For instance, we may choose $z_k^{(n)} = 0$; the polynomial $p_n(z)$ is the sum of the first $n + 1$ terms of the Taylor development of $f(z)$ about the origin. If we set $f(z) \equiv 1/(1 - z)$, then $f(z)$ is analytic in the extended plane except at the point $z = 1$, yet the sequence $p_n(z)$ converges only for $|z| < 1$.

Another instructive example was given by Méray [1884]. Choose $f(z) \equiv 1/z$, and let the points $z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}$ be the $(n + 1)$-st roots of unity. We verify directly that the unique polynomial $p_n(z)$ of degree n is z^n, for in each point $z_k^{(n)}$ we have $z^n = 1/z$. The sequence $p_n(z)$ converges to the value zero for all $|z| < 1$, diverges for $|z| > 1$, and converges to the value $f(z)$ only in the single point $z = 1$, which is a point of interpolation for all the polynomials $p_n(z)$.

If the points (7) are so chosen that $z_k^{(n)} = z_k$ does not depend (for $n \geqq k - 1$) on n, the polynomials $p_n(z)$ and $p_{n+1}(z)$ coincide in the points $z_1, z_2, \cdots, z_{n+1}$. Hence the difference $p_{n+1}(z) - p_n(z)$, a polynomial of degree $n + 1$, must be a constant multiple of the polynomial $(z - z_1)(z - z_2) \cdots (z - z_{n+1})$. This

reasoning is valid even if the points $z_1, z_2, \cdots, z_{n+1}$ are not all distinct. The polynomial $p_n(z)$ is then the sum of the first $n + 1$ terms of a series of the form

$$(8) \qquad f(z) = a_0 + a_1(z - z_1) + a_2(z - z_1)(z - z_2) + \cdots ;$$

this development is to be considered merely as formal for the present. A series (8) is called *Newton's series, a series of interpolation*, or *an interpolation series* in distinction to the more general *sequence of polynomials $p_n(z)$ of interpolation* considered in connection with the unrestricted table (7).

The coefficients in the formal development (8) can be found in succession by an obvious method. For simplicity we still assume $f(z)$ analytic in the points z_k. Set $z = z_1$ and determine a_0:

$$a_0 = f(z_1).$$

Set now $z = z_2$ and determine a_1:

$$a_1 = \frac{f(z_2) - f(z_1)}{z_2 - z_1}, \qquad \text{if} \quad z_2 \neq z_1;$$

if $z_2 = z_1$, differentiate (8) formally and then set $z = z_1 = z_2$:

$$a_1 = f'(z_1).$$

In general, let $a_0, a_1, \cdots, a_{n-1}$ be known, and let precisely m of the points $z_1, z_2, \cdots, z_n$ be equal to z_{n+1}. Differentiate (8) formally m times and set $z = z_{n+1}$. The new quantity by which a_n is multiplied is the value for $z = z_{n+1}$ of the m-th derivative of

$$(z - z_1)(z - z_2) \cdots (z - z_n)$$

and is different from zero, whereas the new quantities by which $a_{n+1}, a_{n+2}, \cdots$ are multiplied all vanish. Thus the coefficient a_n can be computed in terms of $a_0, a_1, \cdots, a_{n-1}$, hence in terms of the numbers $z_1, z_2, \cdots, z_{n+1}$ and the values of derivatives of $f(z)$ at those points z_k.

According to this method of determining the coefficients in the formal expansion (8), the sum of the first $n + 1$ terms of (8) is a polynomial of degree n which coincides with $f(z)$ in the points $z_1, z_2, \cdots, z_{n+1}$, distinct or not. Indeed, it is clear from the method of determining the numbers a_k that the function $f(z) - a_0$ vanishes in the point $z = z_1$, that the function $f(z) - a_0 - a_1(z - z_1)$ vanishes in the points $z = z_1$ and $z = z_2$, and (to proceed by induction) that if the function

$$f(z) - a_0 - a_1(z - z_1) - \cdots - a_{n-1}(z - z_1)(z - z_2) \cdots (z - z_{n-1})$$

vanishes in the points $z_1, z_2, \cdots, z_n$, distinct or not, then the function

$$f(z) - a_0 - a_1(z - z_1) - \cdots - a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

vanishes in the points $z_1, z_2, \cdots, z_{n+1}$, distinct or not.

The two methods of obtaining the formal development (8), by use of the polynomials of interpolation $p_n(z)$ or directly by successive substitution $z = z_1, z_2, \cdots$

(with proper allowance for multiple points z_k) are then equivalent, for according to both methods the sum of the first $n + 1$ terms of (8) is the unique (Theorem 2) polynomial of degree n coinciding with $f(z)$ in the points $z_1, z_2, \cdots, z_{n+1}$. The actual expression for $p_n(z)$ in terms of the values and derivatives of $f(z)$ is called *Newton's interpolation formula.*

A formula for the coefficients a_n in (8) is readily obtained; we have from (4),

$$p_n(z) - p_{n-1}(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$$
$$= \frac{1}{2\pi i}\int_C \frac{(z - z_1)(z - z_2) \cdots (z - z_n)f(t)\,dt}{(t - z_1)(t - z_2) \cdots (t - z_n)(t - z)} - \frac{1}{2\pi i}\int_C \frac{(z - z_1) \cdots (z - z_{n-1})f(t)\,dt}{(t - z_1) \cdots (t - z_{n+1})(t - z)},$$

for z interior to a contour or set of mutually exterior contours C containing the z_k, the function $f(z)$ being analytic on and within C. Thus we find

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t - z_1) \cdots (t - z_{n+1})}.$$

In numerical computation, series of interpolation of polynomials have the great advantage over sequences of polynomials that the determination of the polynomial $p_n(z)$, when the polynomial $p_{n-1}(z)$ is known, requires the determination of the single constant a_n instead of $n + 1$ constants, the coefficients of the powers of z in $p_n(z)$.

If a series of form (8) actually converges to $f(z)$ in the points z_k (for $z = z_k$ the series is a finite sum), and if the z_k are all distinct, then the coefficients a_k must be those of the formal expansion of $f(z)$. If a series of form (8) converges to $f(z)$ in the points z_k not necessarily distinct, and if suitable derived series converge to the corresponding derivatives of $f(z)$ in the multiple points z_k (this condition is surely satisfied if (8) converges to $f(z)$ uniformly on a point set which contains the z_k in its interior), then the series (8) must be the formal expansion of $f(z)$.

The formal expansion (8) of a given function $f(z)$ surely converges to $f(z)$ in the points z_k, and may also converge to $f(z)$ in various other points, as we shall see.

One fundamental problem of interpolation is the representation or expansion of a given function, and another is the study of the existence of a function $f(z)$ (perhaps with certain auxiliary conditions) which takes on prescribed values in given points. If functional values are given in the points (7) or in the points $z_1, z_2, \cdots$, with the usual convention for multiple points, then the sequence of polynomials $p_n(z)$ or the series (8) can be determined whether $f(z)$ exists or not. The convergence of the sequence $p_n(z)$ or the series (8) may be a condition (necessary or sufficient or both) for the existence of the function $f(z)$ and the sequence or series may then actually exhibit the unknown function. This is familiar if all of the z_k lie at the origin, but is true in various other cases; compare §§3.4 and 3.5.

§3.3. Lemniscates and the Jacobi series

A *lemniscate* is a locus of the form

(9) $$\Gamma: |p(z)| = \mu > 0, \qquad p(z) \equiv (z - \beta_1)(z - \beta_2) \cdots (z - \beta_\lambda),$$

where μ and the β_k are fixed, and z varies in accordance with (9); the β_k are not necessarily all distinct. If the β_k are fixed and μ varies, equation (9) represents a *family* of lemniscates; through an arbitrary finite point z_0 of the plane distinct from the β_k passes one and only one lemniscate of the family, namely the lemniscate $|p(z)| = |p(z_0)|$. We shall now develop certain properties of the locus (9), as well as properties of the family.

The lemniscate (9) is the image of the circle $|w| = \mu$ under the transformation $w = p(z)$, so (9) consists of an analytic arc in the neighborhood of each point z, unless we have $p'(z) = 0$. In the neighborhood of a point z of (9) for which $p'(z)$ has an m-fold zero, the locus (9) consists of $m + 1$ analytic arcs passing through the point z with equally spaced tangents. In the neighborhood of every point of (9) there are points z where $|p(z)| > \mu$ and also points z where $|p(z)| < \mu$. When z approaches β_k, the modulus $|p(z)|$ approaches zero; when z becomes infinite, the modulus $|p(z)|$ becomes infinite. Every Jordan arc joining a point β_k to infinity must cut the locus (9).

It follows that the lemniscate (9) consists of a finite number of finite Jordan curves which have a totality of no more than a finite number of intersections, roots of $f'(z)$. Every point β_k lies interior to one such Jordan curve, and (Principle of Maximum) lies interior to only one such curve. No finite region whose boundary belongs to (9) can (Principle of Maximum) have a point of (9) in its interior, and no such region can fail to have a point β_k in its interior. Every point z interior to a finite region whose boundary points belong to (9) satisfies the inequality $|p(z)| < \mu$, and every finite point in the infinite region bounded by (9) satisfies the inequality $|p(z)| > \mu$. The locus (9) consists of a finite number of mutually exterior Jordan curves unless points for which $p'(z) = 0$ lie on the locus. In this case the locus (9) consists of a finite number of Jordan curves which are mutually exterior except for such points.

From the monotonic character of the loci it follows that when μ is sufficiently small, the lemniscate (9) consists of one oval surrounding each of the points β_k; if there are precisely j distinct points β_k, such a lemniscate consists of precisely j ovals. When μ increases, these ovals increase in size, in the sense that the locus $|p(z)| = \mu' > \mu$ is exterior to the locus (9) and the locus (9) is interior to the locus $|p(z)| = \mu'$. When μ increases and the variable locus crosses through an m-fold root of $p'(z)$, its components decrease in number by precisely m; when the variable locus crosses through roots of $p'(z)$ of total multiplicity m, the components of the locus decrease in number by precisely m. When μ is sufficiently large, the locus (9) consists of precisely one Jordan curve. No more than $j - 1$ lemniscates of the family have multiple points.

If the region $|p(z)| > \mu$ is of connectivity q, then every region $|p(z)| > \mu_1 > \mu$ is of connectivity q or less. If n roots of $p'(z)$ lie exterior to (9) and none on (9), each point counted according to its multiplicity, then the locus (9) consists of precisely $n + 1$ Jordan curves.

A further geometric remark is appropriate. All roots of $p'(z)$ lie in the smallest convex point set which contains on or within it the roots of $p(z)$ (Lucas). Hence

any lemniscate Γ which lies entirely exterior to this point set must consist of a single analytic Jordan curve. Some other geometric properties of loci (9) and also of equipotential loci of Green's function have recently been developed by the present writer [1935].

Lemniscates are intimately connected with polynomial expansions:

THEOREM 3. *Let the function $f(z)$ be analytic on and within the lemniscate $\Gamma:|\,p(z)\,| = \mu > 0$, $p(z) = (z - \beta_1)(z - \beta_2) \cdots (z - \beta_\lambda)$; then interior to Γ the function $f(z)$ can be expanded in a series of polynomials of which the n-th term is a polynomial of degree $\lambda - 1$ multiplied by the $(n - 1)$-st power of $p(z)$. The sum $S_n(z)$ of the first n terms of the series coincides with $f(z)$ in the points $\beta_1, \beta_2, \cdots, \beta_\lambda$, each counted of multiplicity n. Moreover, for z on the set $C:|\,p(z)\,| \leqq \mu_1 < \mu$, we have*

$$|\,f(z) - S_n(z)\,| \leqq M_1\mu_1^n/\mu^n, \tag{10}$$

where M_1, independent of n and z, is suitably chosen.

The lemniscate Γ may consist of several mutually exterior contours. The function $f(z)$ defined on and within one such contour need have no monogenic relation to the functions $f(z)$ defined on and within the other contours.

We define $S_n(z)$ as the polynomial of degree $\lambda n - 1$ which coincides in the points β_j (each counted of multiplicity n) with $f(z)$. Then the polynomial $S_n(z) - S_{n-1}(z)$ is of degree $\lambda n - 1$ and has $n - 1$ roots in each of the points β_j, so is the product of $[p(z)]^{n-1}$ by a polynomial of degree $\lambda - 1$. That is, the formal expansion of $f(z)$ can be written

$$f(z) = q_1(z) + q_2(z)\, p(z) + q_3(z)\, [p(z)]^2 + \cdots, \tag{11}$$

where $q_j(z)$ is a polynomial of degree $\lambda - 1$.

We have by (4),

$$f(z) - S_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{[p(z)]^n f(t)\, dt}{[p(t)]^n(t - z)}, \qquad z \text{ interior to } \Gamma. \tag{12}$$

For z on C and t on Γ we have $|\,p(z)\,| \leqq \mu_1 < \mu = |\,p(t)\,|$, which implies convergence as stated in (10).

Series (11) seems first to have been studied by Jacobi [1856], but there have been many more recent uses of that series, for instance by Hilbert [1897], Kienast [1906], and Montel [1910].

§3.4. An analogous series of interpolation

A series of interpolation which possesses many of the properties of (11) may be found under the same hypothesis as follows. Let $S_1^{(j)}(z)$ be the polynomial of degree $j - 1$, $0 < j \leqq \lambda$, which coincides with $f(z)$ in the points $\beta_1, \beta_2, \cdots, \beta_j$; let $S_2^{(j)}(z)$ be the polynomial of degree $j + \lambda - 1$ which coincides with $f(z)$ in the points $\beta_1, \beta_2, \cdots, \beta_j, \beta_1, \beta_2, \cdots, \beta_\lambda$; in general let $S_n^{(j)}(z)$ be the polynomial

of degree $j + (n - 1)\lambda - 1$ which coincides with $f(z)$ in the points $\beta_1, \beta_2, \cdots, \beta_j$ each counted of multiplicity n and in the points $\beta_{j+1}, \beta_{j+2}, \cdots, \beta_\lambda$ each counted of multiplicity $n - 1$. The convergence of $S_n^{(j)}(z)$ to $f(z)$ for z interior to Γ can be proved just as before. The equation analogous to (12) is

$$f(z) - S_n^{(j)}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{(z - \beta_1)(z - \beta_2) \cdots (z - \beta_j)[p(z)]^{n-1} f(t)\, dt}{(t - \beta_1)(t - \beta_2) \cdots (t - \beta_j)[p(t)]^{n-1}(t - z)},$$

z interior to Γ,

and we have as before*

(13) $$|f(z) - S_n^{(j)}(z)| \leqq M^{(j)}(\mu_1/\mu)^n, \qquad z \text{ on } C,$$

where $M^{(j)}$ is independent of n and z.

In our present notation, the sequence $S_n(z)$ of §3.3 is the sequence $S_n^{(\lambda)}(z)$.

We denote now by $p_n(z)$ the $(n + 1)$-st term of the sequence

(14) $$S_1^{(1)}(z), S_1^{(2)}(z), \cdots, S_1^{(\lambda)}(z), S_2^{(1)}(z), S_2^{(2)}(z), \cdots, S_2^{(\lambda)}(z), S_3^{(1)}(z), S_3^{(2)}(z), \cdots,$$

so that $p_n(z)$ is the polynomial of degree n which coincides with $f(z)$ in the first $n + 1$ points of the sequence

$$\beta_1, \beta_2, \cdots, \beta_\lambda, \beta_1, \beta_2, \cdots, \beta_\lambda, \beta_1, \beta_2, \cdots.$$

Thus $p_n(z)$ is the sum of the first $n + 1$ terms of the series of polynomials corresponding to the sequence (14), which is a series of interpolation:

(15) $$\begin{aligned} f(z) = a_0 &+ a_1(z - \beta_1) + a_2(z - \beta_1)(z - \beta_2) + \cdots \\ &+ a_\lambda(z - \beta_1) \cdots (z - \beta_\lambda) + a_{\lambda+1}(z - \beta_1) \cdots (z - \beta_\lambda)(z - \beta_1) \\ &+ a_{\lambda+2}(z - \beta_1) \cdots (z - \beta_\lambda)(z - \beta_1)(z - \beta_2) + \cdots. \end{aligned}$$

This series possesses interesting properties entirely analogous to the properties of Taylor's series, which is the special case that all the β_j coincide, whether λ is unity or greater than unity.

THEOREM 4. *If the function $f(z)$ is single-valued and analytic throughout the interior of the lemniscate Γ': $|p(z)| = \mu'$, $p(z) = (z - \beta_1)(z - \beta_2) \cdots (z - \beta_\lambda)$, but is not single-valued and analytic throughout the interior of any lemniscate $|p(z)| = \mu'' > \mu'$, then the formal development* (15) *of $f(z)$ found by interpolation in the points β_j converges to $f(z)$ interior to Γ', uniformly on any closed point set interior to Γ', and diverges exterior to Γ'. Moreover, we have*

(16) $$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1/(\mu')^{1/\lambda},$$

(17) $$|f(z) - p_n(z)| \leqq M(\mu_1/\mu)^{n/\lambda}, \textit{ for } z \textit{ on } C\colon |p(z)| \leqq \mu_1 < \mu',$$

* Inequality (13) can also be proved from Theorem 3 by virtue of the remark concerning equation (20) below.

where M is independent of n and z but not of μ_1 and μ, where $p_n(z)$ is the sum of t first $n + 1$ terms of (15), *and where $\mu > \mu_1$ is an arbitrary positive number less than μ*

If there be given an arbitrary function $f(z)$ analytic in the points β_i, the exists a greatest number μ' (finite or infinite) such that $f(z)$ is single-valued a analytic interior to $\Gamma' : |p(z)| = \mu'$. Otherwise expressed, Γ' has the proper that the function $f(z)$ can be uniquely extended analytically along paths interi to Γ' from the β_j to each point interior to Γ'; but this property holds for lemniscate $|p(z)| = \mu'' > \mu'$. The function $f(z)$ defined in the neighborhoo of one of the points β_i need have no relation to the analytic extensions of t function $f(z)$ defined in the neighborhoods of the other points β_i. Either has on it a singularity of $f(z)$, or Γ' has at least one multiple point and the a alytic extensions of $f(z)$ to that point from the various points β_i along paths i terior to Γ' do not coincide, or both of these eventualities occur.

The greatest number μ' exists, for $f(z)$ is surely analytic interior to some le niscate $|p(z)| = \mu$; if μ is sufficiently small, the lemniscate consists of small co tours in the neighborhoods of the points β_i, where $f(z)$ is known to be analyti If $f(z)$ is single-valued and analytic interior to each of the lemniscates $|p(z$ $= m_k$, where $m_1 < m_2 < m_3 < \cdots \to \mu'$, then $f(z)$ is single-valued and analyt interior to the lemniscate $|p(z)| = \mu'$, for $f(z)$ can be analytically extended fro the β_j along a path interior to $\Gamma' : |p(z)| = \mu'$ to any particular point z interi to Γ'; such a point z lies interior to some lemniscate $|p(z)| = m_k$. Such extension along paths interior to Γ' must be unique, for any two paths interi to Γ' also lie interior to some lemniscate $|p(z)| = m_k$. We proceed with t proof of Theorem 4.

Inequality (13) can be written in the following form:

$$|f(z) - p_n(z)| \leq M^{(j)}(\mu_1/\mu)^m = M^{(j)}(\mu_1/\mu)^{(n+\lambda-j+1)/\lambda} \leq M^{(j)}(\mu_1/\mu)^{n/\lambda}, \qquad z \text{ on } \text{(}$$

where $n = j + (m - 1)\lambda - 1$, $0 < j \leq \lambda$, and this implies (17) if M is chosen the greatest of the λ quantities $M^{(j)}$. Inequality (17) includes the assertion Theorem 4 relative to the convergence of (15) interior to Γ'.

Let us rewrite (15) in the form

$$f(z) = \sum_{n=0}^{\infty} a_n q_n(z).$$

By inspection of the sequence $q_{m\lambda+j}(z)$ for fixed j, $0 < j \leq \lambda$, $m = 0, 1, 2, \cdots$ we have for fixed z

$$\lim_{m\to\infty} |q_{m\lambda+j}(z)|^{1/(m\lambda+j)} = |p(z)|^{1/\lambda},$$

and this limit is approached uniformly for z on any lemniscate Γ. Indeed, i is not a point β_i, we have

$$\log |q_{m\lambda+j}(z)|^{1/(m\lambda+j)} = \frac{1}{m\lambda + j}\log|(z - \beta_1)\cdots(z - \beta_j)| + \frac{m}{m\lambda + j}\log|p(z)$$

which justifies the remark. The remark holds for every j, so we have

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = |p(z)|^{1/\lambda},$$

for every z, uniformly on any lemniscate Γ: $|p(z)| = \mu$.

It is perhaps most satisfactory in the study of (16) to prove a general theorem:

THEOREM 5. *Let us suppose that the limit*

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = |q(z)| \tag{18}$$

exists on some point set C and let us introduce the notation

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1/A,$$

where the a_n are arbitrary, and where A is finite or infinite. Then the series $\sum a_n q_n(z)$ converges at all points of C at which $|q(z)| < A$, uniformly at all points of C at which $|q(z)| \leqq q < A$ and at which (18) *holds uniformly, and diverges at all points of C at which $|q(z)| > A$.*

For points z on the set C at which $|q(z)| \leqq q < A$ and at which (18) holds uniformly (choose q_1 and q_2, $q < q_1 < q_2 < A$), we have uniformly for n sufficiently large

$$|q_n(z)|^{1/n} \leqq q_1 > q, \qquad |a_n|^{1/n} \leqq 1/q_2 > 1/A.$$

Then for n sufficiently large we have uniformly

$$|a_n q_n(z)| \leqq (q_1/q_2)^n, \qquad q_1 < q_2,$$

so the given series converges uniformly. This proof yields also convergence at an individual point of C at which $|q(z)| < A$.

For a point z on the set $|q(z)| = q > A$ (choose q_1 and q_2, $q > q_1 > q_2 > A$), we have for n sufficiently large

$$|q_n(z)|^{1/n} \geqq q_1 < q.$$

For an infinity of indices n we have

$$|a_n|^{1/n} \geqq 1/q_2 < 1/A.$$

Then for an infinity of indices n the inequality

$$|a_n q_n(z)| \geqq (q_1/q_2)^n, \qquad q_1 > q_2,$$

is valid, so the general term of the given series does not approach zero, the series cannot converge, and Theorem 5 is established.

We add the following

COROLLARY. *Suppose the functions $q_n(z)$ of Theorem 5 are polynomials, that the function $q(z)$ can be chosen some positive power of a polynomial, and that* (18) *is valid uniformly on each locus $|q(z)| = q$ corresponding to a set of numbers q everywhere dense, $0 < q < \infty$. Then the series $\sum a_n q_n(z)$ converges at all points at*

which $|q(z)| < A$, *uniformly on any closed set* $|q(z)| \leqq \alpha < A$, *and diverges at* *points at which* $|q(z)| > A$ *and at which* (18) *is valid.*

Uniform convergence of the given series on a locus $|q(z)| = q$ implies unifo convergence on and within that locus (a lemniscate if $q \neq 0$). Theorem 5 sho that the given series converges uniformly on any locus $|q(z)| = q < A$, wher belongs to the favored set; if $\alpha < A$ is given there exists such a q, $\alpha < q <$ and this completes the proof of the corollary.

Under the conditions of the corollary, there is consequently a *lemniscate* *convergence* γ: $|q(z)| = A$, which degenerates to a finite set of points if $A =$ and includes the entire plane if $A = \infty$. The given series converges interior γ, uniformly on any closed set interior to γ, and diverges at all points exterior γ at which (18) is valid.

For the application of this Corollary to series (15) of Theorem 4, we set $q(z)$ $p(z)^{1/\lambda}$. Then (15) converges at all points interior to some locus $|p(z)| =$ (a lemniscate if $\mu_2 \neq 0$, $\mu_2 \neq \infty$), uniformly on any interior closed set, and verges at all exterior points. We have already proved that (15) converges all points interior to $|p(z)| = \mu'$. Series (15) can converge uniformly on lemniscate $|p(z)| = \mu'' > \mu'$, for $f(z)$ is not single-valued and analytic throug out the interior of any such lemniscate. Hence Γ': $|p(z)| = \mu'$ is the lemn cate of convergence of (15), so (16) follows from Theorem 5, and Theorem 4 completely proved.

Special types of series (15) have been considered by various writers, but t general series (15) seems first to have been studied by Kienast [1906]; compa also Martinotti [1910]. Some further properties of such series, particularly the lemniscate of convergence, are obtained by Walsh and Curtiss [1935].

§3.5. A more general series of interpolation

It seems reasonable to suppose, from the treatment just given of the particul series of interpolation (15), that broadly speaking the convergence of an arl trary series of interpolation

$$f(z) = a_0 + a_1(z - z_1) + a_2(z - z_1)(z - z_2) + \cdots \tag{19}$$

depends not on the entire sequence of points z_k, but merely on the asymptot character of the sequence z_k. We shall shortly make this remark more preci (Theorem 6), but meanwhile a general fact regarding the arbitrary series (19) of importance.

The series

$$\frac{f(z) - a_0 - a_1(z - z_1) - \cdots - a_{k-1}(z - z_1) \cdots (z - z_{k-1})}{(z - z_1)(z - z_2) \cdots (z - z_k)} \tag{20}$$
$$= a_k + a_{k+1}(z - z_{k+1}) + a_{k+2}(z - z_{k+1})(z - z_{k+2}) + \cdots$$

is also a series of interpolation, and if the coefficients $a_0, a_1, \cdots, a_{k-1}$ have bee determined from (19) by interpolation, then formal determination of the coef cients $a_k, a_{k+1}, \cdots$ by interpolation from (20) is equivalent to formal determin

tion of those coefficients from (19). The coefficient a_ν ($\nu \geqq k$) is found from (19) by differentiating (19) a number of times m equal to the number of times $z_{\nu+1}$ appears in the set $z_1, z_2, \cdots, z_\nu$, and then setting $z = z_{\nu+1}$. The coefficient a_ν is found from (20) by differentiating (20) a number of times n equal to the number of times $z_{\nu+1}$ appears in the set $z_{k+1}, z_{k+2}, \cdots, z_\nu$, and then setting $z = z_{\nu+1}$. In the latter case we have divided by $(z - z_1)(z - z_2) \cdots (z - z_k)$, which contains $(z - z_{\nu+1})$ as a factor precisely $m - n$ times. In each case, whether in (19) or (20), the coefficient a_ν is so chosen that the expression

$$f(z) - a_0 - a_1(z - z_1) - a_2(z - z_1)(z - z_2) - \cdots$$
$$- a_\nu(z - z_1)(z - z_2) \cdots (z - z_\nu)$$

has a zero of order at least $m + 1$ in the point $z = z_{\nu+1}$, and this condition defines a_ν uniquely (§3.2) in terms of the coefficients $a_0, a_1, \cdots, a_{\nu-1}$. Thus the successive determination of the coefficients $a_k, a_{k+1}, \cdots$ from (20) is equivalent to the determination of those coefficients from (19).

The great advantage of (20) over (19) in the proof of the convergence of (19) is that the points of interpolation $z_1, z_2, \cdots, z_k$ no longer appear explicitly in (20), yet convergence of the one series implies convergence of the other except in the points $z = z_j$, $j \leqq k$, so it is clear that the convergence or divergence of (19) for large classes of functions $f(z)$ depends only on the ultimate behavior of the points of the sequence z_ν. If the function $f(z)$ is analytic at a point or in a region, the left-hand member of (20) is also analytic at that point or in that region, provided of course a suitable definition of that left-hand member is supplied in the points $z_1, z_2, \cdots, z_k$.

The transformation (20) of series (19) is to be used in proving a generalization of Theorem 4:

THEOREM 6. *Let the sequence $\beta_1', \beta_2', \cdots$ be asymptotic to the sequence*

$$\beta_1, \beta_2, \cdots, \beta_\lambda, \beta_1, \beta_2, \cdots, \beta_\lambda, \beta_1, \beta_2, \cdots$$

in the sense

$$\lim_{\nu\to\infty} \beta'_{\nu\lambda+1} = \beta_1, \quad \lim_{\nu\to\infty} \beta'_{\nu\lambda+2} = \beta_2, \cdots, \quad \lim_{\nu\to\infty} \beta'_{\nu\lambda+\lambda} = \beta_\lambda .$$

If the function $f(z)$ is defined in all the points β_j' and satisfies the hypothesis of Theorem 4, then the formal development

$$f(z) = a_0 + a_1(z - \beta_1') + a_2(z - \beta_1')(z - \beta_2') + \cdots \tag{21}$$

found by interpolation in the points β_j', converges to $f(z)$ interior to Γ', uniformly on any closed point set interior to Γ', and diverges exterior to Γ' except in the points β_j' exterior to Γ'. Moreover, we have (16) *and* (17) *satisfied, where the a_n are now the coefficients in* (21) *and $p_n(z)$ is now the sum of the first $n + 1$ terms of* (21).

It is convenient in the proof of Theorem 6 to study the λ different subsequences chosen from (21) analogous to the λ subsequences $S_n^{(j)}(z)$ considered in §3.4.

The treatment is similar for all the λ sequences, so we take up in detail a singl sequence, the analogue of the Jacobi sequence of Theorem 3.

Let $\mu < \mu'$ and $\mu_1 < \mu$ be arbitrary, and let us choose μ_2, μ_3, and μ_4, such tha we have $\mu_1 < \mu_2 < \mu_3 < \mu_4 < \mu'$, $\mu_2/\mu_3 = \mu_1/\mu$; our notation is the same as that c Theorem 4. For t on Γ: $|p(z)| = \mu_4$ we have uniformly

$$(22)\qquad \lim_{\nu\to\infty} |(t - \beta'_{\nu\lambda+1})(t - \beta'_{\nu\lambda+2}) \cdots (t - \beta'_{(\nu+1)\lambda})| = |p(t)| = \mu_4,$$

and in particular for suitably large ν we have

$$(23)\qquad |(t - \beta'_{\nu\lambda+1})(t - \beta'_{\nu\lambda+2}) \cdots (t - \beta'_{(\nu+1)\lambda})| > \mu_3.$$

For z on C': $|p(z)| = \mu_1$ we have uniformly

$$\lim_{\nu\to\infty} |(z - \beta'_{\nu\lambda+1})(z - \beta'_{\nu\lambda+2}) \cdots (z - \beta'_{(\nu+1)\lambda})| = |p(z)| = \mu_1,$$

so for suitably large ν we have

$$(24)\qquad |(z - \beta'_{\nu\lambda+1})(z - \beta'_{\nu\lambda+2}) \cdots (z - \beta'_{(\nu+1)\lambda})| < \mu_2.$$

We now make the transformation of (21) which corresponds to (20); we writ (20) in the form

$$(25)\qquad F(z) = a_{N\lambda} + a_{N\lambda+1}(z - \beta'_{N\lambda+1}) + a_{N\lambda+2}(z - \beta'_{N\lambda+1})(z - \beta'_{N\lambda+2}) + \cdots$$

where N is so large that for $\nu \geqq N$ inequalities (23) and (24) are valid for t on and z on C'. If $S_n(z)$ is the polynomial of degree $\lambda n - 1$ which is the sum of tl first λn terms of the right-hand member of (25), we have for z on C',

$$F(z) - S_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z - \beta'_{N\lambda+1}) \cdots (z - \beta'_{(N+n)\lambda})}{(t - \beta'_{N\lambda+1}) \cdots (t - \beta'_{(N+n)\lambda})}\frac{F(t)\,dt}{(t - z)},$$

$$(26)\qquad |F(z) - S_n(z)| \leqq M_1(\mu_2/\mu_3)^n = M_1(\mu_1/\mu)^n,$$

where M_1 is suitably chosen.

We prefer to express (26) in terms of $f(z)$ and the original sequence of (21 The function $(z - \beta'_1)(z - \beta'_2) \cdots (z - \beta'_{N\lambda})$, where N is fixed as above, is limite on C, so we have whenever $(m + 1)/\lambda$ is an integer greater than N,

$$(27)\qquad |f(z) - p_m(z)| \leqq M'_1(\mu_1/\mu)^{-N+(m+1)/\lambda} = M'(\mu_1/\mu)^{m/\lambda}, \qquad z \text{ on } C',$$

where $p_m(z)$ denotes the sum of the first $m + 1$ terms of the right-hand memb of (21). This inequality clearly holds whenever $(m + 1)/\lambda$ is an integer, wheth greater than N or not, if we permit a suitable modification of M'_1 and M'.

Inequality (27) can readily be proved (with possible modification in the co stant M') for the indices m corresponding to each of the remaining $\lambda - 1$ s quences analogous to $S_n(z)$, and hence (27) holds for *every* m and for z on ($|p(z)| \leqq \mu_1$. Thus (17) is established for Theorem 6.

As a general remark, which is entirely independent of the present situation, v notice that the relation

$$\lim_{n\to\infty} |g_n(z)| = |g(z)| \neq 0 \tag{28}$$

implies the relation

$$\lim_{n\to\infty} |g_1(z)\cdot g_2(z) \cdots g_n(z)|^{1/n} = |g(z)|, \tag{29}$$

provided the $g_k(z)$ are bounded from zero; and we notice that if (28) is valid uniformly and the $g_k(z)$ are uniformly bounded from zero, then (29) is also valid uniformly. The proof is immediate if we make use of the fact that convergence or uniform convergence of a sequence implies convergence or uniform convergence of the sequence of Cesàro means; equations (28) and (29) are equivalent respectively to the equations

$$\lim_{n\to\infty} \log |g_n(z)| = \log |g(z)|,$$

$$\lim_{n\to\infty} \frac{\log |g_1(z)| + \log |g_2(z)| + \cdots + \log |g_n(z)|}{n} = \log |g(z)|.$$

We write (21) in the form $f(z) = \sum_{n=0}^{\infty} a_n q_n(z)$. By virtue of (22) and the remark just made, we have

$$\lim_{m\to\infty} |q_{m\lambda}(z)|^{1/m} = |p(z)|,$$

and this implies (the proof is easy if logarithms are considered)

$$\lim_{n\to\infty} |q_n(z)|^{1/n} = |p(z)|^{1/\lambda},$$

uniformly on any Γ: $|p(z)| = \mu > 0$ or on any closed arc of any Γ which does not pass through a point β_n'. Exterior to any lemniscate Γ there are at most a finite number of points β_n'. It follows then from the Corollary to Theorem 5 that a series (21) where the a_k are arbitrary converges at all points interior to a certain lemniscate $|p(z)| = \mu$ ($\mu = 0$ or ∞ not excluded) and diverges at all points other than the β_n' exterior to this locus. The reasoning used in connection with Theorem 4 can now be applied directly; this proves (16) and completes the proof of Theorem 6.

It is remarkable that in Theorem 6 any values whatever can be assigned to $f(z)$ in the points β_n' which lie on or exterior to Γ', and the series (21) converges to those values in such points and to $f(z)$ interior to Γ'; this is true even if $f(z)$ has the lemniscate Γ' as a natural boundary. In particular, we may choose $f(z)$ as different from zero in points β_n' on or exterior to Γ' and identically zero interior to Γ'. Then we obtain a series (21) which represents the function zero interior to Γ' but whose coefficients do not all vanish; an explicit example is given in §8.2.

Let the points β_n' satisfy the requirements of Theorem 6. If $f(z)$ is given single-valued and analytic interior to a lemniscate Γ'': $|p(z)| = \mu''$, then one may care to choose values other than those of $f(z)$ for interpolation at a finite

number (but not all) of the points β_n' interior to Γ''. If this is done, le be the smallest of the numbers $|p(\beta_n')|$ for the points β_n' in which values ot than those of $f(z)$ are used for interpolation. Then the lemniscate Γ': $|p$ $= \mu'$ is the lemniscate of convergence of (21), for if (21) is written in the fo (25), where N is sufficiently large, the function $F(z)$ is single-valued and a lytic interior to Γ' but not interior to any $|p(z)| = \mu''' > \mu'$.

Let the points β_n' still satisfy the hypothesis of Theorem 6. A necess and sufficient condition that a series of form (21) where the a_n are arbitr should converge at even a single point different from the β_k' is that $\overline{\lim}|a_n$ be finite. If this condition is satisfied, the series converges at all points terior to some lemniscate Γ', and the series is the unique formal expans (found by interpolation in all the β_k') of the function represented.

The situation of Theorem 6 is of particular interest in the case $\lambda = 1$. Se (21) then has properties quite similar to those of Taylor's series, and is of inte in connection with both the problems of interpolation mentioned at the end §3.2. If $f(z)$ is given analytic in the neighborhood of the point β_1, the series (converges to $f(z)$ in some neighborhood of β_1. In particular, two distinct fu tions $f(z)$ analytic at β_1 cannot have the same values in the points β_j' (all disti or not), for they have the same formal development, and that development valid in some neighborhood of β_1. If the functional values are given in the poi β_j' (distinct or not), Theorem 6 may be used to give a necessary and suffici condition (in terms of either the convergence of the series (21) or condition (1 for the existence of a function $f(z)$ analytic in a given neighborhood of β_1 taki on the prescribed values. Moreover, the exact radius of convergence of (2 may be found from (16), and this yields information concerning the singularit of $f(z)$ if the latter exists.

This special case $\lambda = 1$ of (21) has been studied by Bendixson [1886] a others.

CHAPTER IV

DEGREE OF CONVERGENCE OF POLYNOMIALS. OVERCONVERGENCE

§4.1. Equipotential curves in conformal maps

The equipotential curves for Green's function for an infinite region with pole at infinity are similar in character to the lemniscates that we described in §3.3:

THEOREM 1. *Let C be a closed limited point set of the z $(= x + iy)$-plane whose complement K (with respect to the extended plane) is connected and regular in the sense that K possesses a Green's function $G(x, y)$ with pole at infinity. Then the function $w = \phi(z) = e^{G+iH}$, where H is conjugate to G in K, maps K conformally but not necessarily uniformly onto the exterior of the unit circle γ in the w-plane so that the points at infinity in the two planes correspond to each other; interior points of K correspond to exterior points of γ, and exterior points of γ correspond to interior points of K.*

Each equipotential locus in K such as C_R: $G(x, y) = \log R > 0$, or $|\phi(z)| = R > 1$, either consists of a finite number of finite mutually exterior analytic Jordan curves or consists of a finite number of contours which are mutually exterior except that each of a finite number of points may belong to several contours.

Green's function $G(x, y)$ with pole at infinity for the region K is the unique function which is harmonic in K except at infinity, which outside of some circle C' can be expressed as $\log (x^2 + y^2)^{1/2}$ plus some function harmonic exterior to C' and approaching a finite value at infinity; and which is continuous in the closed region $\bar{K}$ except at infinity and vanishes on the boundary.

The function $\phi(z)$ is uniquely determined except for a multiplicative constant of modulus unity. When K is simply connected that constant is usually chosen so that we have $\phi'(\infty) > 0$.

If K is multiply connected, the function $w = \phi(z)$ cannot set up a one-to-one continuous correspondence between the points of K and the points exterior to γ, so $w = \phi(z)$ cannot be single-valued in K. The branch points of $\phi(z)$ can have no limit point interior to K, as we shall shortly prove. The modulus of $\phi(z)$ is e^G, which is single-valued in K.

If K is the complement of an arbitrary closed limited point set consisting of more than a single point, and if K is simply connected, then K is necessarily regular; indeed, this follows from the existence of the mapping function $w = \phi(z)$. More generally,* the complement K of C is regular if it is connected and of finite connectivity provided C is closed and limited and has no isolated points.

* See for instance Lebesgue [1907] or Osgood [1912].

By successive mapping of K using the ordinary results on the mapping of a *simply con-*

If C is the point set $|z| \leqq r$, the function $\phi(z)$ is z/r, and the locus C_R is circle $|z| = rR$. If C is the point set $-1 \leqq z \leqq 1$, z real, the mapping func w is given by $z = \frac{1}{2}(w + 1/w)$, and the locus C_R is the ellipse whose foci are points $+1$ and -1, and whose semimajor axis is $\frac{1}{2}(R + 1/R)$.

If K is simply connected, it follows directly from the conformal map C_R is the transform of the circle $|w| = R$, and hence is an analytic Jor curve. Moreover, the locus C_R always lies interior to the locus $C_{R'}$, $R' >$ The situation is not quite so simple if K is multiply connected.

In our discussion of the more general loci C_R, we make frequent use of fact that a function harmonic but not identically constant in a region can h no maximum nor minimum in that region, and if identically constant in s subregion is identically constant in the entire given region.

A *critical point* of $G(x, y)$ is a point (interior to K) where both first par derivatives of $G(x, y)$ vanish, or if we prefer, a point where $\phi'(z)$ vanishes; it be noted that the vanishing of the derivative of any particular branch of $\phi(z)$ point $z = z'$ implies the vanishing of the derivative of every branch of $\phi(z$ $z = z'$. There are at most a finite number of zeros of $\phi'(z)$, hence at mos finite number of critical points of $G(x, y)$ in any closed region in K; any l point of critical points of $G(x, y)$ cannot be infinite and must lie on C.

Through an arbitrary point (x_0, y_0) of K passes one and only one locus namely, the locus $G(x, y) = G(x_0, y_0)$. In the neighborhood of such a point, locus C_R is the image of the circle $|w| = R$ under the transformation $w = \phi$ and hence C_R consists of a single analytic arc if $\phi'(x_0 + iy_0) \neq 0$ and of m - branches through (x_0, y_0) with equally spaced tangents if (x_0, y_0) is an m-f root of $\phi'(z)$ (i.e., an m-fold critical point of $G(x, y)$). In the neighborhood every point of C_R there are points (x, y) of K where $G(x, y) > \log R$ and a points (x, y) where $G(x, y) < \log R$.

When (x, y) approaches C, the function $G(x, y)$ approaches zero; when $(x,$ becomes infinite, the function $G(x, y)$ becomes infinite. Every Jordan arc jo ing C to infinity must cut each C_R. We can now conclude that C_R consists o finite number of finite Jordan curves which have a totality of no more than finite number of intersections. Every point of C lies interior to one and o one such Jordan curve of a given C_R. No finite region whose boundary longs to C_R can have a point of C_R in its interior, and no such region can fail have at least one component of C in its interior. Every point (x, y) of interior to a finite region whose boundary belongs to C_R satisfies the inequal $G(x, y) < \log R$, and every finite point in an infinite region interior to whose finite boundary points belong to C_R satisfies the inequality $G(x, y)$ $\log R$. The locus C_R consists of a finite number of mutually exterior Jord curves unless critical points of $G(x, y)$ lie on the locus. In the latter case t locus C_R consists of a finite number of Jordan curves which are mutua exterior except for such critical points.

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nected region, we can map K onto a region bounded by analytic Jordan curves; comp §4.3. Then other well known methods [Bieberbach, 1927; Kellogg, 1929] can be applied prove the regularity of K.

It follows from the monotonic character of the loci that when R is sufficiently small the locus C_R consists of one contour surrounding each component of C, provided K is of finite connectivity. Whether K is of finite or infinite connectivity, when R increases the contours of C_R increase in size, in the sense that C_R is always interior to $C_{R'}$ if $R < R'$. When R increases and the variable locus crosses through an m-fold critical point of $G(x, y)$, its components decrease in number by precisely m; when the variable locus crosses through critical points of $G(x, y)$ of total multiplicity m, the components of the locus decrease in number by precisely m. When R is sufficiently large, the locus C_R consists of precisely one contour.

The point set $G(x, y) > \log R$ has C_R as its boundary and hence must be a region. If this region is of connectivity q, then the region $G(x, y) > \log R' > \log R$ is of connectivity q or less. If m critical points of $G(x, y)$ lie exterior to C_R and none on C_R, each point counted according to its multiplicity, then C_R consists of precisely $m + 1$ Jordan curves. The region $G(x, y) > R$ is always exterior to C_R, and the point set $G(x, y) < R$ is always interior to C_R.

If K is of connectivity q, the point at infinity considered an interior point, then no C_R can consist of more than q contours. The function $G(x, y)$ has precisely $q - 1$ critical points, counted according to their multiplicities. No more than $q - 1$ loci C_R have multiple points. If K is of infinite connectivity, the function $G(x, y)$ has a countably infinite number of critical points, the loci $C_{R_1}, C_{R_2}, \cdots$ which pass through critical points of $G(x, y)$ are countably infinite in number, and R_n approaches unity as n becomes infinite.

The loci C_R will be of much importance in the present chapter and the succeeding ones. We shall consistently use the notation C_R; even if a limited point set C consists of a number of Jordan curves or other point sets which separate the plane, if the exterior of C is denoted by K, then the function $w = \phi(z)$ which maps K onto the exterior of the unit circle in the w-plane $(\phi(\infty) = \infty)$ leads directly to loci $|\phi(z)| = R > 1$ in K which we denote as before by C_R. We add for reference several other remarks concerning the loci C_R.

If the function $w = \phi(z)$ maps K onto the exterior of γ so that the points at infinity in the two planes correspond to each other, the function $w = \phi(z)/R$ similarly maps the exterior of C_R onto the exterior of γ. That is to say, *the locus* $[C_R]_{R'}$ *is the locus* $|\phi(z)/R| = R'$, *or the locus* $C_{RR'}$.

If C is bounded by a lemniscate and is the point set C: $|p(z)| \leqq \mu > 0$, $p(z) = (z - z_1)(z - z_2) \cdots (z - z_\lambda)$, then the function $\phi(z)$ can be chosen as $[p(z)/\mu]^{1/\lambda}$, and the curve C_R is the lemniscate $|p(z)| = \mu R^\lambda$.

Let C' be a closed proper subset of the set C of Theorem 1 and let $G'(x, y)$ be Green's function with pole at infinity for the complement K' (supposed connected and regular) of C'. Let C_R' denote the locus $G'(x, y) = \log R > 0$. *Then* C_R' *lies interior to* C_R. The function $G'(x, y) - G(x, y)$ is harmonic in K, has only a removable singularity at infinity, is non-negative at each boundary point of K, and at some boundary points of K (namely those boundary points of K not boundary points of K') is actually positive. Then the function

$G'(x, y) - G(x, y)$ is positive at every (interior) point of K. At every po of C_R, we have $G' - G = G' - \log R > 0$, so the value of $G'(x, y)$ is grea than $\log R$. That is to say, the locus C'_R lies interior to C_R.

We add a further remark relative to the critical points of the function $G(x,$ of Theorem 1, which may be proved [Walsh, 1935] by the use of Lucas's theor (§3.3) and the methods of §4.2: *All critical points of $G(x, y)$ lie in the small convex point set which contains C.* Consequently, any locus C_R which lies tirely exterior to that convex point set must consist of a single contour. T locus $|\phi(z)| = R$, where R is greater than the maximum of $|\phi(z)|$ on the boun ary of that convex point set, therefore consists of a single contour.

In the particular case that the point set C of Theorem 1 is bounded by a fin number of mutually exterior analytic Jordan curves, the function $G(x, y)$ can extended harmonically across those curves into the interior of C; this follov after a conformal map of any of those curves onto the axis of reals, by Schwar principle of reflection. We proceed to prove that the function $G(x, y)$ can ha no critical point on the boundary of C; that is to say, for K the interior norm derivative $\partial G/\partial n$ is always *greater* than zero on C. On C we have $\partial G/\partial s =$ so the vanishing of $\partial G/\partial n$ on C would imply a critical point of $G(x, y)$ on In the neighborhood of a critical point of $G(x, y)$ the locus $G = 0$ consists several branches with equally spaced tangents. No point of the locus $G =$ can lie exterior to C, and the boundary of C can have no corner; hence tl boundary of C passes through no critical point.

§4.2. Approximation of Jordan curves by a lemniscate

We shall apply the results of §§3.3 and 3.4 in the study of approximation l polynomials on quite general point sets, and in order to carry out that applicati it will be convenient to make use of

THEOREM 2. *A finite number of arbitrary mutually exterior analytic Jord curves of the finite z-plane can be uniformly approximated by the same lemniscat that is to say, given a point set C which consists of the mutually exterior analyt Jordan curves $K_1, K_2, \cdots, K_\nu$, and a number $\eta > 0$ such that the distance betwe any two curves K_i and K_j is greater than 2η, then there exists a lemniscate* Γ *$|(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_k)| = \mu > 0$ exterior to C which consists of ν mutual exterior contours each containing in its interior precisely one of the curves K_j, su that no point of Γ is at a distance from C greater than η.*

Let $G(x, y)$ be Green's function with pole at infinity for the complement K C, and suppose in the neighborhood of the point at infinity we have $G(x, y) =$ $\log (x^2 + y^2)^{1/2} + G_0(x, y)$, where $G_0(x, y)$ approaches the value $-g$ as (x, y) becomes infinite. We shall prove that *for the function* $V(x, y) \equiv G(x, y) +$

$$(1) \qquad V(x, y) = \int_C \phi_1(s) \log r \, ds, \qquad \phi_1(s) = \frac{1}{2\pi} \frac{\partial V(x, y)}{\partial n},$$

n being the *exterior* normal for C, where $r = |z - \zeta|$, $ds = |d\zeta|$, and $z = x + i$ is an arbitrary point of K.

Let C' denote a circle whose center is P: (x, y) which contains C in its interior. We have

$$(2)\quad G(x, y) = \frac{1}{2\pi}\int_C \left(\log r \frac{\partial G}{\partial n} - G \frac{\partial \log r}{\partial n}\right) ds + \frac{1}{2\pi}\int_{C'} \left(\log r \frac{\partial G}{\partial n} - G \frac{\partial \log r}{\partial n}\right) ds,$$

where as before $r = | z - \zeta |$, $ds = | d\zeta |$, and n is the interior normal for the region bounded by C' and C; equation (2) is valid even though C' depends on (x, y). In the second integral on the right-hand side we make the substitution $G = \log r + G_1$. Even though r is not $| \zeta |$, the function $G_1(\xi, \eta)$ approaches the value $-g$ as $\zeta = \xi + i\eta$ becomes infinite. By Gauss's mean value theorem applied to the exterior of C' we have

$$g = \frac{1}{2\pi}\int_{C'} G_0 \frac{\partial \log r}{\partial n}\, ds = \frac{1}{2\pi}\int_{C'} G_1 \frac{\partial \log r}{\partial n}\, ds,$$

and we have also

$$\int_{C'} \log r \frac{\partial G_1}{\partial n}\, ds = 0.$$

If we use the fact that G vanishes on C, we now have

$$G(x, y) + g = \frac{1}{2\pi}\int_C \log r \frac{\partial(G + g)}{\partial n}\, ds,$$

which is equivalent to (1).

Let $K'_1, K'_2, \cdots, K'_\nu$ be mutually exterior Jordan curves in K which contain in their respective interiors the curves $K_1, K_2, \cdots, K_\nu$, such that no point of K'_j is at a distance greater than the prescribed η from K_j; such curves K'_j exist, by §1.3, Corollary to Theorem 2. The function $V(x, y)$ is continuous on all the K'_j, and on all those curves takes on a minimum value $g_1 > g$. Each point of the K'_j lies on or exterior to the locus $G(x, y) = g_1 - g$. Let us choose ϵ, $0 < \epsilon < (g_1 - g)/2$. The locus $V = g + \epsilon$ is the locus $G(x, y) = \epsilon$, which therefore (§4.1) consists of ν contours γ_j, lying in the respective rings $K_jK'_j$. Similarly, the locus $V = g_1 - \epsilon$ is the locus $G(x, y) = g_1 - g - \epsilon$, which consists of ν contours γ'_j lying in the respective rings $\gamma_j K'_j$.

Let us introduce the notation $u_0 = \int_C \phi_1(s)\, ds$. The function $\phi_1(s)$ is *positive* on C, by §4.1, since it is except for the factor $(2\pi)^{-1}$ the normal derivative of Green's function for a region bounded by a finite number of *analytic* Jordan curves. We can introduce a new variable

$$u(\zeta) = \int_0^{s(\zeta)} \phi_1(s)\, ds$$

defined at all points of C, and $u(\zeta)$ varies monotonically from 0 to u_0 when s varies monotonically from 0 to the total length of C. If u is suitably defined on C, the one-to-one-ness of the correspondence between points of C and points of

the interval $0 \leqq u \leqq u_0$ fails in at most a finite number of points of C or that interval. Then by (1) we can write (this new n has no connection with t previous n)

$$V(x, y) = \int_0^{u_0} \log r \, du,$$

(3) $$V(x, y) = \lim_{n \to \infty} \frac{u_0}{n} (\log r_1 + \log r_2 + \cdots + \log r_n),$$

where $r_1, r_2, \cdots, r_n$ are the distances from z to n points ζ_m of C which correspo to the n equidistant values $u_0/n, 2u_0/n, \cdots, u_0$ of u in the interval $(0, u_0)$. T points ζ_m depend of course on n, but for simplicity that dependence is not in cated in the notation.

Equation (3) is valid uniformly on any closed limited point set K' interior K, for on K the function $\log r$ is continuous and hence uniformly continuo We can write

(4) $$\begin{aligned} V(x, y) &- \frac{u_0}{n} (\log r_1 + \log r_2 + \cdots + \log r_n) \\ &= \sum_{m=0}^{n-1} \int_{mu_0/n}^{(m+1)u_0/n} (\log r - \log r_{m+1}) \, d \end{aligned}$$

The right-hand member is in absolute value not greater than the maximum

(5) $$u_0 \,|\log r - \log r_{m+1}|$$

for z on K', for $m = 0, 1, \cdots, n - 1$, and for ζ on the portion of C represent by

$$mu_0/n \leqq u \leqq (m + 1)u_0/n.$$

To be sure, it may occur that ζ and ζ_m lie on different curves K_j, but that ca happen for at most ν terms of the right-hand member of (4), terms whose su can be made uniformly as small as we please in absolute value. If we exclud those ν possible terms, the quantity (5) can be made as small as we please un formly for all z on K', and hence (3) is valid uniformly on K'.

Let us now choose λ, $g + \epsilon < \lambda < g_1 - \epsilon$, and choose $\epsilon' > 0$, $g + \epsilon < \lambda -$ $\epsilon' < \lambda + \epsilon' < g_1 - \epsilon$. Choose N so large that for z in the closed rings $\gamma_j\gamma_j'$, w have

$$\left| V(x, y) - \frac{u_0}{N} \log (r_1 r_2 \cdots r_N) \right| < \epsilon'.$$

Denote by Γ the locus $(u_0/N) \log (r_1 r_2 \cdots r_N) = \lambda$, which is a lemniscate

$$|(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_N)| = e^{\lambda N/u_0};$$

we shall prove that Γ is the lemniscate desired.

Any Jordan arc (in the ring $\gamma_j\gamma_j'$) connecting a point of γ_j with a point of γ must intersect Γ, for on γ_j we have $(u_0/N) \log (r_1 r_2 \cdots r_N) < g + \epsilon + \epsilon' < \lambda$

and on γ_j' we have $(u_0/N)\log(r_1r_2\cdots r_N) > g_1 - \epsilon - \epsilon' > \lambda$. At a point (x, y) of Γ in a ring $\gamma_j\gamma_j'$, we have $g + \epsilon < \lambda - \epsilon' < V(x, y) < \lambda + \epsilon' < g_1 - \epsilon$, so Γ cannot intersect γ_j or γ_j'. The points ζ_m all lie on C. Then Γ cannot (§3.3) consist partly of a contour in $\gamma_j\gamma_j'$ which does not contain γ_j in its interior, cannot consist partly of more than one contour in any ring $\gamma_j\gamma_j'$, cannot consist partly of a contour interior to any γ_j or of any contour exterior to all the rings $\gamma_j\gamma_j'$, and cannot have a multiple point. The lemniscate Γ consists precisely of ν contours; the j-th contour lies in the ring $\gamma_j\gamma_j'$ and contains γ_j in its interior. The proof is complete.

Theorem 2 is due to Hilbert [1897] in the case $k = 1$ and to Walsh and Russell [1934] in the general case.

§4.3. Approximation of modulus of the mapping function

The formulas already used can be somewhat simplified, for we have

$$u_0 = \int_C \phi_1(s)\,ds = \frac{1}{2\pi}\int_C \frac{\partial V}{\partial n}\,ds = -\frac{1}{2\pi}\int_{C'} \frac{\partial \log r}{\partial n}\,ds = 1.$$

Then equation (3), which is valid uniformly on any closed limited point set interior to K, can also be written in the form

$$(6)\qquad \lim_{n\to\infty} |\omega_n(z)|^{1/n} = e^g |\phi(z)|, \qquad \omega_n(z) = (z - \zeta_1^{(n)})(z - \zeta_2^{(n)})\cdots(z - \zeta_n^{(n)}),$$

where the dependence on n of the points $\zeta_m^{(n)}$ is now indicated. Equation (6) is of the precise form we shall require for later application (Chapters VII, VIII, IX).

For (x, y) interior to a curve K_j, the right-hand member of (3) represents the function

$$\frac{1}{2\pi}\int_C \log r\,\frac{\partial G}{\partial n}\,ds = \frac{1}{2\pi}\int_C\left(\log r\,\frac{\partial G}{\partial n} - G\,\frac{\partial \log r}{\partial n}\right)ds$$
$$= \frac{1}{2\pi}\int_{C'}\left(\log r\,\frac{\partial G}{\partial n} - G\,\frac{\partial \log r}{\partial n}\right)ds = g,$$

where n is now the exterior normal for C'. Thus we have

$$\lim_{n\to\infty} |\omega_n(z)|^{1/n} = e^g,$$

interior to C, uniformly on any closed set interior to C. This too is a result which will find later application.

Theorem 2 in its present form can clearly be used to prove the corresponding result where the Jordan curves K_j are no longer analytic but are the most general Jordan curves; it is sufficient to denote the given curves by C and to apply Theorem 2 as proved to a suitable locus C_R; indeed this reasoning extends Theorem 2 even to the case that C is the boundary of an arbitrary closed limited point set whose complement K is connected and regular. The extension of equation (6),

however, even to the case that the K_j are arbitrary Jordan curves, is more difficult. To be sure, all the discussion given of both Theorem 2 and equation (6) is valid if each curve K_j consists of a finite number of arcs each with continuous curvature. But our proof of (6) requires substantial modification to apply to the case that the K_j are arbitrary Jordan curves. The result itself persists:

THEOREM 3. *Let the point set C consist of a finite number of mutually exterior Jordan curves $K_1, K_2, \cdots, K_\nu$ of the finite z-plane, let K be the exterior of C, and let $w = \phi(z)$ map K onto the exterior of $|w| = 1$ so that the points at infinity in the two planes correspond to each other. Then there exists a set of points $\zeta_m^{(n)}$, $m = 1, 2, \cdots, n$; $n = 1, 2, \cdots$, which lie on C and are such that we have*

$$(7) \qquad \lim_{n\to\infty} |\omega_n(z)|^{1/n} = e^g |\phi(z)|, \qquad \omega_n(z) = (z - \zeta_1^{(n)})(z - \zeta_2^{(n)}) \cdots (z - \zeta_n^{(n)}),$$

uniformly on any closed limited point set in K, where g is a suitably chosen constant.

If K is mapped conformally and one-to-one onto a region Q of the τ-plane bounded by analytic Jordan curves $Q_1, Q_2, \cdots, Q_\nu$ so that the points at infinity correspond to each other, then the points $\zeta_m^{(n)}$ can be chosen on the K_j so as to correspond to points on the Q_j which divide the totality of curves Q_j into n equal parts, equal not necessarily with respect to arc length but with respect to increments of the function

$$u(\tau) = \frac{1}{2\pi} \int_0^{s(\tau)} \frac{\partial G'}{\partial n}\, ds,$$

where G' is Green's function for Q with pole at infinity.

The given region K can be mapped conformally and uniformly onto a region Q bounded by analytic Jordan curves, for we can map uniformly in succession onto the exterior of a circle the exterior of K_1, the exterior of the transform of K_2 under the transformation just made, the exterior of the transform of K_3 under the transformation just made, and so on. The j-th of these maps carries K_j or its transform into an analytic Jordan curve, and the succeeding maps carry the successive transforms of K_j also into analytic curves. The ν-th transform of K is the region Q desired. We can suppose that the points at infinity in the two regions K and Q correspond to each other.

Let K be transformed onto Q by the transformation $z = \omega(\tau)$, $\tau = \alpha + i\beta$. As a matter of convenience we assume, as we may do, that $\omega'(\tau)$ has the value unity at infinity:

$$\lim_{\tau\to\infty} |z/\tau| = \lim_{\tau\to\infty} |\omega(\tau)/\tau| = |\omega'(\tau)|_{\tau=\infty} = 1, \qquad \lim_{\tau\to\infty} [\log|z| - \log|\tau|] = 0.$$

Then Green's function $G(x, y)$ for K with pole at infinity is the transform of Green's function $G'(\alpha, \beta)$ for Q with pole at infinity. Consequently the two numbers g (notation of §4.2) for the two regions K and Q are equal.

In the τ-plane we denote the points of subdivision of the curves Q_j by $\tau_m^{(n)}$, so we have the analogue of (6):

$$\lim_{n\to\infty} | (\tau - \tau_1^{(n)})(\tau - \tau_2^{(n)}) \cdots (\tau - \tau_n^{(n)}) |^{1/n} = e^{g} | \phi(z) | ,$$

uniformly on any closed limited point set in Q, where τ and z are related by the transformation $z = \omega(\tau)$. In order to establish Theorem 3, it remains then to prove

$$(8) \lim_{n\to\infty} \left| \frac{[\omega(\tau) - \omega(\tau_1^{(n)})][\omega(\tau) - \omega(\tau_2^{(n)})] \cdots [\omega(\tau) - \omega(\tau_n^{(n)})]}{(\tau - \tau_1^{(n)})(\tau - \tau_2^{(n)}) \cdots (\tau - \tau_n^{(n)})} \right|^{1/n} = 1, \zeta_m^{(n)} = \omega(\tau_m^{(n)}),$$

uniformly on any closed limited point set in Q.

The logarithm of the left-hand member of (8) can be written

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \left| \frac{\omega(\tau) - \omega(\tau_j^{(n)})}{\tau - \tau_j^{(n)}} \right|,$$

and this can clearly be written

$$(9) \qquad \int_0^{u_0} \log \left| \frac{\omega(\tau) - \omega(\sigma)}{\tau - \sigma} \right| du = \frac{1}{2\pi} \int_{Q_1 + Q_2 + \cdots + Q_\nu} \log \left| \frac{\omega(\tau) - \omega(\sigma)}{\tau - \sigma} \right| \frac{\partial G'}{\partial n} | d\sigma | ,$$

where σ is the variable point on the Q_j and where τ is a point interior to Q.

The function $[\omega(\tau) - \omega(\sigma)]/(\tau - \sigma)$ is a non-vanishing analytic function of σ interior to Q (except for a removable singularity at τ), when τ is held fast interior to Q, and this function is continuous in the closed region $\bar{Q}$. The logarithm of the modulus of this function is harmonic interior to Q, continuous in $\bar{Q}$. Then by a familiar formula of potential theory,* the right-hand member of (9) represents the value of the harmonic function considered as a function of σ at the pole of Green's function $G'(\alpha, \beta)$. This familiar formula is valid even when (as here) the pole of Green's function lies at infinity. Thanks to our requirement that $\omega'(\sigma)$ should be unity for $\sigma = \infty$, we have

$$\lim_{\sigma\to\infty} \frac{\omega(\tau) - \omega(\sigma)}{\tau - \sigma} = 1 ,$$

which implies (8) uniformly as indicated, and our theorem.

* The usual formula of potential theory (where G is Green's function for the region bounded by C)

$$(a) \qquad u(x, y) = \frac{1}{2\pi} \int_C u \frac{\partial G}{\partial n} ds$$

may be proved from (2), where in (2) the function G is now replaced by $u(x, y)$, but (a) is ordinarily proved only in case $u(x, y)$ is harmonic in the *closed* region $\bar{Q}$ bounded by C, even when (as here) C consists of a finite number of analytic curves. It is sufficient, however, if $u(x, y)$ is harmonic interior to Q, continuous in $\bar{Q}$, and this may be proved 1) by studying the equation analogous to (a), where the integral is taken over the locus $G = \log R > 0$, and by allowing R to approach unity, or 2) by expanding $u(x, y)$ in $\bar{Q}$ according to the methods of Chapter II in a suitable uniformly convergent series of harmonic rational functions; the analogue of equation (a) is valid for each term of the series, hence for the function $u(x, y)$; compare §2.6.

The description in Theorem 3 of the choice of the points $\zeta_m^{(n)}$ is of course independent of the particular region Q chosen. In the case $\nu = 1$, we may take Q_1 as the unit circle and may choose the $\zeta_m^{(n)}$ as the transforms of the n-th roots of unity.

Another method for the proof of (7) is to be developed in §7.6.

Theorem 3 was proved by Fejér [1918] in the important case $\nu = 1$.

§4.4. Approximation of modulus of mapping function, continued

Theorem 3 is valid, together with its proof, if some or all of the components K_j of the boundary of C are Jordan arcs instead of Jordan curves. But there are difficulties in extending that theorem to still more general point sets C, for then the mapping function $\omega(\tau)$ may fail to be continuous. Nevertheless, an analogous result can be established:

THEOREM 4. *Let C be a closed limited point set of the z-plane whose complement K is connected and regular (in the sense that K possesses a Green's function with pole at infinity). Then there exists a set of points $\zeta_m^{(n)}$, $m = 1, 2, \cdots, n$; $n = 1, 2, \cdots$ in K with no limit point except on C such that* (7) *is valid uniformly on any closed limited point set in K.*

It will be noted that the function $e^g \mid \phi(z) \mid$ which appears in the right-hand member of (7) is the modulus of a function which maps K onto the exterior of a circle $\mid w \mid = r = e^g$ so that the points at infinity correspond to each other, and so that the derivative at infinity of the mapping function has the modulus unity; the number r is, moreover, uniquely determined. Indeed, we can write $e^g \cdot \phi'(\infty) = \lim_{z\to\infty} e^g\phi(z)/z$, and we have in the notation of §4.2

$$\log \mid e^g\phi(z)/z \mid = g + G(x, y) - \log (x^2 + y^2)^{1/2} = G_0(x, y) + g;$$

the function $G_0(x, y) + g$ is harmonic at infinity and has the value zero there. As a consequence of our present interpretation of the function $e^g \mid \phi(z) \mid$, it follows that if the closed interior of any C_R is allowed to take the rôle previously taken by C, then the function in the right-hand member of (7) is unaltered, for the function $e^g \mid \phi(z) \mid$ is continuous on C_R, and $w = e^g\phi(z)$ maps the exterior of C_R onto the exterior of the circle $\mid w \mid = e^gR$ so that the points at infinity in the two planes correspond to each other; the derivative at infinity of the mapping function has the modulus unity. The function which maps the exterior of C_R onto the exterior of $\mid w \mid = 1$ is of course $w = \phi(z)/R$, so when the closed interior of C_R takes the rôle previously taken by C, the new number e^gR takes the rôle previously taken by e^g.

This number $e^g = 1/\mid \phi'(\infty) \mid$ is known as the *capacity*, or *Robin's constant*, or the *transfinite diameter* of either the boundary of K or the complement of K. This concept appears frequently in the sequel.

Let the number $R_0 > 1$ be arbitrary, and let us choose a sequence R_l, $R_0 > R_1 > R_2 > \cdots \to 1$, in such a way that C_{R_l} consists of a number of mutually exterior analytic Jordan curves. The points $\xi_m^{(n)}$ can be chosen on C_{R_1} so that for every $n > N_1$ we have

$$\big|\, | \omega_n(z) |^{1/n} - e^g \,| \phi(z) | \,\big| < 1, \qquad z \text{ on or exterior to } C_{R_0}, \ |z| \leqq 1,$$

where $\omega_n(z) = (z - \xi_1^{(n)})(z - \xi_2^{(n)}) \cdots (z - \xi_n^{(n)})$. The number e^g and the function $\phi(z)$ refer to the mapping of K. Similarly, the points $\xi_m^{(n)}$ can be chosen on C_{R_1} so that for $n > N_2 > N_1$ we have

$$\big|\, | \omega_n(z) |^{1/n} - e^g \,| \phi(z) | \,\big| < 1/2, \qquad z \text{ on or exterior to } C_{R_1}, \ |z| \leqq 2.$$

In general, the points $\xi_m^{(n)}$ can be chosen on C_{R_l} so that for $n > N_l > N_{l-1}$ we have

$$\big|\, | \omega_n(z) |^{1/n} - e^g \,| \phi(z) | \,\big| < 1/l, \qquad z \text{ on or exterior to } C_{R_{l-1}}, \ |z| \leqq l.$$

Points $\zeta_m^{(n)}$ chosen as follows now satisfy the requirements of Theorem 4. For values of n not greater than N_2, the points $\zeta_m^{(n)}$ shall be the points $\xi_m^{(n)}$ just considered on C_{R_1}. For values of n greater than N_2 but not greater than N_3, the points $\zeta_m^{(n)}$ shall be the points $\xi_m^{(n)}$ just considered on C_{R_2}. For values of n greater than N_l but not greater than N_{l+1}, the points $\zeta_m^{(n)}$ shall be the points $\xi_m^{(n)}$ just considered on C_{R_l}. The numbers $\zeta_m^{(n)}$ are thereby determined for all values of n, for we have $N_1 < N_2 < N_3 < \cdots$.

The capacity of the set C_{R_l} is $R_l e^g$, which approaches e^g. It is therefore clear from the corresponding discussion (§4.3) for analytic curves that these points $\zeta_m^{(n)}$ can be chosen so that (7) is valid uniformly on any closed limited point set in K and so that we have simultaneously

$$\lim_{n\to\infty} | \omega_n(z) |^{1/n} = e^g$$

interior to C, uniformly on any closed set interior to C.

A slight modification of the proof just given enables one to include in the totality of points $\zeta_m^{(n)}$ an arbitrary set interior to K having no limit point in the extended plane except perhaps on C.

Without going into the details of the proof, we remark that under broad conditions on the point set C, Green's function $G_k(x, y)$ for the complement of a suitably chosen variable point set $C^{(k)}$ interior to C approaches Green's function $G(x, y)$ for the complement of C; compare §2.1, Theorem 2 for the case that K is simply connected. This fact may be employed in a manner analogous to the proof of Theorem 4, to determine points $\zeta_m^{(n)}$ *interior to* C such that (7) is valid uniformly on any closed limited point set in K.

If C is composed of a finite number of mutually exterior *analytic* Jordan curves, the function $G(x, y)$ can be extended harmonically into the interior of C. A suitable point set C' interior to C is the interior of a locus $G(x, y) = -\epsilon < 0$, and by Theorem 3 points $\zeta_m^{(n)}$ can be chosen on C' (hence interior to C) such that (7) is valid uniformly on any closed limited point set exterior to C'.

§4.5. Degree of convergence. Sufficient conditions

We shall now apply the results of §4.2 in the proof of

THEOREM 5. *Let C be a closed limited point set whose complement K is connected and regular. If the function $f(z)$ is single-valued and analytic on and within*

C_R, there exists a sequence of polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \cdots$ such that we have

$$(10) \qquad |f(z) - p_n(z)| \leq M/R^n, \qquad z \text{ on } C,$$

where M depends on R, but not on n or z.

The function $f(z)$ is single-valued and analytic on and within C_R, hence (§1.5, Theorem 7) is single-valued and analytic on and within some $C_{R'}$, $R' > R$, and we can suppose the locus $C_{R'}$ to have no multiple points. There exists some lemniscate Γ containing C in its interior which lies interior to $C_{R'/R}$; in fact such a lemniscate exists (Theorem 2) between $C_{(R+R')/(2R)}$ and $C_{R'/R}$. The curve $\Gamma: |p(z)| = \mu_1, p(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_k)$, lies interior to $C_{R'/R}$, so the curve Γ_R: $|p(z)| = \mu_1 R^k$ lies interior to $[C_{R'/R}]_R = C_{R'}$. Then $f(z)$ is single-valued and analytic on and within Γ_R. By §3.4, Theorem 4, there exist polynomials $p_n(z)$ of respective degrees n such that we have (10) valid for z on Γ. But inequality (10), valid on Γ, is also valid on C interior to Γ, so Theorem 5 is established.

Theorem 5 is due to Faber [1903] (implicitly), Bernstein [1912], Szegö [1921] (explicitly), and Walsh [1926b] in the cases respectively that C is the closed interior of an analytic curve, a line segment, the closed interior of a Jordan curve, or an arbitrary closed limited point set whose complement is simply connected, and is due to Walsh and Russell [1934] in the general case.

In the special case that C is a circle, Theorem 5 follows directly from Cauchy's inequality for the coefficients in the Taylor development of $f(z)$.

We have in Theorem 5 a new proof of §1.6, Theorem 8, for the case of approximation by polynomials; this proof is a generalization of Hilbert's original proof [1897] for approximation in a Jordan region. The present Theorem 5 is clearly more specific than the previous result of §1.6.

An obvious lack of completeness in Theorem 5 as just proved is that the polynomials $p_n(z)$ depend on R, whereas R is not in any way uniquely determined from $f(z)$ and C. We shall prove in §5.1 from Theorem 5 the existence of polynomials $p_n(z)$ which satisfy the conditions of Theorem 5 and which depend on $f(z)$ and C but not on the number R. Another proof of the existence of such polynomials $p_n(z)$ can also be given by the use of Theorem 4, as we shall now indicate.

Under the hypothesis of Theorem 5, let $p_n(z)$ be the polynomial of degree n $(n = 0, 1, 2, \cdots)$ which coincides with $f(z)$ in the points $\zeta_1^{(n+1)}, \zeta_2^{(n+1)}, \cdots, \zeta_{n+1}^{(n+1)}$ of Theorem 4. The function $f(z)$ need not be defined in all of those points, but the totality of points $\zeta_k^{(n+1)}$ has no limit point except on C, so $f(z)$ is surely defined in $\zeta_k^{(n+1)}$ if n is sufficiently large. We have

$$(11) \qquad f(z) - p_n(z) = \frac{1}{2\pi i} \int_{C'} \frac{\omega_{n+1}(z)\, f(t)\, dt}{\omega_{n+1}(t)\, (t - z)}, \qquad z \text{ interior to } C',$$

where C' is an arbitrary set of mutually exterior contours on and within which $f(z)$ is analytic, provided that C' contains the points $\zeta_k^{(n+1)}$ in its interior. The polynomial $p_n(z)$ is independent of C' under these conditions.

Let now $f(z)$ be single-valued and analytic on and within C_R; then $f(z)$ is also single-valued and analytic on and within some $C_{R'}$, $R' > R$. We suppose the locus $C_{R'}$ to have no multiple points. Choose R_1, $1 < R_1 < R'/R$, and choose R_2, $RR_1 < R_2 < R'$. Choose C' as the locus $C_{R'}$.

It follows from (7) that we have uniformly for n sufficiently large

$$| \omega_{n+1}(z) | \leqq (c^\rho R_2/R)^{n+1}, \qquad z \text{ on } C_{R_1},$$

$$| \omega_{n+1}(t) | \geqq (c^\rho R_2)^{n+1}, \qquad t \text{ on } C_{R'}.$$

Then for z on C_{R_1} and hence for z on C we have by (11)

$$(12) \qquad | f(z) - p_n(z) | \leqq M/R^n,$$

provided merely n is sufficiently large, and where M depends on R but not on n or z.

The polynomials $p_n(z)$ may be defined only for n sufficiently large because the function $f(z)$ may be defined in the points $\zeta_k^{(n+1)}$ only for n sufficiently large. For the excluded values of n, let us arbitrarily define $p_n(z)$ as identically zero. Inequality (12), hitherto also proved only for n sufficiently large, will then be valid for z on C for all values of n, provided the number M (naturally depending on R) is appropriately modified. This completes our application of Theorem 4, proof of the existence of polynomials $p_n(z)$ which satisfy (12) but do not depend on R.

If the point set C consists of a finite number ν of mutually exterior closed Jordan regions, we may use the points $\zeta_k^{(n+1)}$ of Theorem 3 instead of the points $\zeta_k^{(n+1)}$ of Theorem 4 in defining the polynomials $p_n(z)$. These polynomials were introduced and studied by Fejér in the case $\nu = 1$, and in that case were used by Szegö to establish (12). It will be noticed that the essential difference between the methods of Hilbert and Fejér for approximation to a given function in a given Jordan region is that Hilbert first approximates the Jordan curve bounding the given region by a lemniscate and then uses the Jacobi series (§3.3) or its equivalent (§3.4) corresponding to the lemniscate, whereas Fejér considers interpolation directly in points on the given Jordan curve, equally distributed (or more generally uniformly distributed, §7.6) with respect to the parameter u.

§4.6. Degree of convergence. Necessary conditions. Overconvergence

In order to study the converse of Theorem 5, it will be convenient to have for reference the

LEMMA. *Let C be a closed limited point set whose complement K is connected and regular. If the polynomial $P(z)$ of degree n satisfies the inequality $| P(z) | \leqq L$ for z on C, then we have*

$$(13) \qquad | P(z) | \leqq LR^n, \qquad z \text{ on or within } C_R.$$

The function $P(z)/[\phi(z)]^n$ (notation of Theorem 1) is analytic exterior to C except possibly for branch points and although the function is not necessarily single-valued its modulus is single-valued exterior to C. This function is analytic even at infinity if suitably defined there. No maximum of the modulus of this

function can occur in the extended plane exterior to C, unless that modulus is everywhere constant. When z approaches C, the modulus can approach no limit greater than L, for the modulus of $\phi(z)$ approaches unity. Thus the inequality $| P(z)/[\phi(z)]^n | \leqq L$ is valid for z exterior to C, and (13) is valid on C_R. The modulus of $P(z)$ can have no proper maximum interior to C_R, so inequality (13), valid for z on C_R, is also valid for z on or within C_R.

The Lemma is due to Bernstein [1912] in the case that C is a line segment, to Faber [1922] in the case that C is bounded by a finite number of contours, and to Walsh [1926b] in the general case; the present method of proof is due to M. Riesz [1916], and is similar to the now usual method of proof of Schwarz's Lemma.

The following theorem is a kind of converse, although not exact, of Theorem 5:

THEOREM 6. *Let C be a closed limited point set whose complement is connected and regular. Let the polynomials $p_n(z)$ satisfy inequality* (10) *for z on C, where M is independent of n and z. Then the sequence $p_n(z)$ converges for z interior to C_R, uniformly on any closed point set interior to C_R. The function $f(z)$ can be extended from C along paths interior to C_R so as to be single-valued and analytic at every point interior to C_R.*

From the inequalities for z on C

$$| f(z) - p_n(z) | \leqq M/R^n, \qquad | f(z) - p_{n+1}(z) | \leqq M/R^{n+1},$$

we have for z on C

$$| p_{n+1}(z) - p_n(z) | \leqq \frac{1}{R^n}\left[M + \frac{M}{R} \right].$$

For z on C_{R_1} we have by the Lemma,

$$| p_{n+1}(z) - p_n(z) | \leqq \frac{R_1^{n+1}}{R^n}\left[M + \frac{M}{R} \right]. \tag{14}$$

Inequality (14) implies the uniform convergence of the sequence $p_n(z)$ on and within every C_{R_1}, $R_1 < R$; by inequality (10) the limit of the sequence $p_n(z)$ on C is $f(z)$; Theorem 6 follows at once.

This theorem was proved in the special case that C is a line segment by S. Bernstein [1912], and in the general case by Walsh [1926b].

Theorems 5 and 6 are concerned entirely with *regions* of analyticity of the function $f(z)$. Quite recently Sewell [1935a] has obtained more specific results for the case that K is simply connected by relating a more precise inequality than (10) to the properties of $f(z)$ on C_R itself.

In Theorem 6 we have supposed for simplicity that inequality (10) is given for all values of n. It is clearly sufficient if that inequality holds merely for n sufficiently large. The conclusion of the theorem holds also, as the reader will easily prove, if (10) is valid for a sequence of polynomials $p_n(z)$ of respective degrees $n = n_1, n_2, \cdots$, with $n_k > n_{k-1}$, provided $n_k - n_{k-1}$ is bounded.

We shall use the term *overconvergence* for the phenomenon illustrated by Theorem 6, namely that certain sequences known to converge sufficiently rapidly on a given point set C necessarily converge on a point set containing C in its interior. The term overconvergence is used by Ostrowski with a somewhat different significance.

§4.7. Maximal convergence

THEOREM 7. *Let C be a closed limited point set whose complement is connected and regular. If the function $f(z)$ is single-valued and analytic on C, there exists a greatest number ρ (finite or infinite) such that $f(z)$ is single-valued and analytic at every point interior to C_ρ. If $R < \rho$ is arbitrary, there exist polynomials $p_n(z)$ of respective degrees $n = 0, 1, 2, \cdots$ such that* (10) *is valid for z on C; but there exist no polynomials $p_n(z)$ such that* (10) *is valid for z on C where $R > \rho$.*

The existence of the number ρ can be proved by the method used in §3.4. The remainder of the theorem follows from Theorem 5 and Theorem 6. Theorem 7 is due to S. Berhstein [1912] in the case that C is a line segment, to Walsh [1926b] in the case that the complement of C is simply connected, and to Walsh and Russell [1934] in the general case A simpler but somewhat less specific theorem, an immediate consequence of Theorems 5 and 6, is:

Let C be a closed limited point set whose complement is connected and regular. A necessary and sufficient condition that $f(z)$ be single-valued and analytic on C is that there exist polynomials $p_n(z)$ of respective degrees n such that (10) *is valid for z on C, where R is some number greater than unity.*

Let the point set C of Theorem 7 consist for definiteness of two mutually exterior Jordan regions C' and C''. Approximation of $f(z)$ on C can be studied by considering the two new problems, of approximating on C the two new functions $f_1(z)$ equal to $f(z)$ on C' and zero on C'', and $f_2(z)$ equal to zero on C' and $f(z)$ on C''. Study of these two new problems (such a study has been made by a number of writers) will obviously yield a sequence of polynomials converging uniformly to $f(z)$ on C like a convergent geometric series, in the sense that an inequality of form (10) is valid with $R > 1$, but may yield less favorable results than Theorem 7 relative to degree of convergence, and hence less favorable relative to regions of overconvergence. We illustrate this remark by choosing an extreme case, that $f(z)$ is an entire function of z Let R_1 be so chosen that the locus C_{R_1} has a double point. Each of the two inequalities

$$|f_1(z) - p_n^{(1)}(z)| \leqq M/R^n, \qquad |f_2(z) - p_n^{(2)}(z)| \leqq M/R^n, \qquad z \text{ on } C,$$

where $p_n^{(1)}(z)$ and $p_n^{(2)}(z)$ are sequences of polynomials of respective degrees n, can be satisfied for an arbitrary $R < R_1$ but neither of them can be satisfied for any $R > R_1$. Nevertheless, the inequality

$$|f(z) - p_n(z)| \leqq M/R^n, \qquad z \text{ on } C,$$

where $p_n(z)$ is a sequence of polynomials of respective degrees n, can be satisfied for an *arbitrary* R.

The polynomials $p_n(z)$ contemplated in Theorem 7, when C and $f(z)$ are given, depend on R. But we have already proved (§4.5) and shall prove again later (§5.1) that under the hypothesis of Theorem 7 on $f(z)$ there exist polynomials $p_n(z)$ independent of R of respective degrees n such that (10) is valid for z on C for *every* $R < \rho$, where M depends on R but not on n or z. Such a sequence of polynomials $p_n(z)$ is said to converge to $f(z)$ on C *maximally*, or *with the greatest geometric degree of convergence*. This concept [Walsh, 1933c, 1935c] is of much importance in the sequel, particularly in Chapters V and VII. It follows from Theorem 6 that such a sequence $p_n(z)$ converges to $f(z)$ at every point interior to C_ρ, uniformly on any closed set interior to C_ρ. The concept has meaning only for approximation to a function $f(z)$ single-valued and analytic on a closed limited point set C whose complement is connected and regular. It is sufficient, however, if the polynomials $p_n(z)$ are not defined for every n but merely for n sufficiently large; the fundamental properties of the sequence persist. We shall consistently use ρ to indicate the number defined in Theorem 7.

Another way of expressing the requirements on the sequence $p_n(z)$ which converges maximally to $f(z)$ on C is

$$\overline{\lim_{n\to\infty}}\, \mu_n^{1/n} = 1/\rho, \qquad \mu_n = \max\,[\,|\,f(z) - p_n(z)\,|\,,\ z \text{ on } C\,];$$

from (10) we have $\overline{\lim}_{n\to\infty}\, \mu_n^{1/n} \leq 1/R$ for every $R < \rho$, and the inequality $\overline{\lim}_{n\to\infty}\, \mu_n^{1/n} < 1/\rho$ is impossible, as follows immediately from Theorem 7.

Inequality (17) of §3.4 expresses essentially the maximal convergence of the sequence $p_n(z)$ involved to the function $f(z)$ on every C: $|\,p(z)\,| \leqq \mu_1 < \mu'$, $\mu_1 > 0$

THEOREM 8. *Let the function $f(z)$ be single-valued and analytic on the closed limited point set C whose complement is connected and regular. A necessary and sufficient condition that a sequence $p_n(z)$ of polynomials of respective degrees n converge to $f(z)$ on C maximally is that the sequence $p_n(z)$ converge to $f(z)$ maximally on the closed interior of every C_σ, $1 < \sigma < \rho$.*

The sufficiency of the condition is immediate, for let $R < \rho$ be arbitrary and choose σ, $1 < \sigma < \rho/R$. The function $f(z)$ is single-valued and analytic interior to $C_\rho = [C_\sigma]_{\rho/\sigma}$, and we have $R < \rho/\sigma$, so for z on and within C_σ we have

$$|\,f(z) - p_n(z)\,| \leqq M/R^n. \tag{15}$$

This inequality, being valid on and within C_σ, is valid for z on C, so the sequence $p_n(z)$ converges to $f(z)$ maximally on C.

The necessity of the condition is also easy to prove. Let $\sigma < \rho$ be arbitrary, $\sigma > 1$, and also $R < \rho/\sigma$; we have $C_\rho = [C_\sigma]_{\rho/\sigma}$ and hence are to prove the va-

lidity of (15) for z on C_σ. From (14) obtained as in the proof of Theorem 6, we have by setting $R_1 = \sigma$ and by replacing R by $\sigma R < \rho$,

$$| p_{n+1}(z) - p_n(z) | \leqq \frac{1}{R^n}\left[\sigma M + \frac{M}{R}\right], \qquad z \text{ on or within } C_\sigma .$$

Let us now write $f(z) = p_m(z) + [p_{m+1}(z) - p_m(z)] + [p_{m+2}(z) - p_{m+1}(z)] + \cdots$ for z on or within C_σ:

$$| f(z) - p_m(z) | = \left| \sum_{k=m}^{\infty} [p_{k+1}(z) - p_k(z)] \right| \leqq \left[\sigma M + \frac{M}{R}\right] \sum_{k=m}^{\infty} \frac{1}{R^k} = \frac{M + \sigma M R}{R^m (R - 1)} .$$

The proof is complete.

It is appropriate to remark that this last method of proof yields the following

COROLLARY. *If* (10) *is valid for z on C, for a particular $R > 1$, then we have*

$$| f(z) - p_n(z) | \leqq M_1(\sigma/R)^n, \qquad z \text{ on } C_\sigma, \qquad \sigma < R,$$

where M_1 is suitably chosen.

Maximal convergence persists in certain cases after differentiation:

THEOREM 9. *Let the function $f(z)$ be single-valued and analytic on the closed limited point set C (not a single point) whose complement is simply connected. If the sequence $p_n(z)$ of polynomials of respective degrees n converges maximally to $f(z)$ on C, then the sequence $p_n'(z)$ converges maximally to $f'(z)$ on C.*

Let $R < \rho$ be arbitrary, and choose σ, $1 < \sigma < \rho/R$. The sequence $p_n(z)$ converges maximally to $f(z)$ on C_σ, so we have as in (15)

$$(16) \qquad | f(z) - p_n(z) | \leqq M/R^n, \qquad z \text{ on } C_\sigma .$$

For z on C we have

$$f'(z) - p_n'(z) = \frac{1}{2\pi i} \int_{C_\sigma} \frac{f(t) - p_n(t)}{(t - z)^2}\, dt .$$

By virtue of (16) we find now

$$| f'(z) - p_n'(z) | \leqq M_1/R^n, \qquad z \text{ on } C,$$

where M_1 does not depend on n or on z.

The polynomial $p_n'(z)$ is of degree $n - 1$. The function $f(z)$ is single-valued and analytic interior to C_ρ but has a singularity on C_ρ, hence the same is true of $f'(z)$. The theorem is established.

Theorem 9 does not extend to the case that the complement of C is multiply connected, as we show by an example. Let C be the closed interior of the lemniscate $| z^2 - 1 | = 1/2$, which consists of two ovals, containing respectively the points 1 and -1. Let $f(z)$ be unity in the right-hand oval and zero in the left-hand oval. The curve C_ρ is $| z^2 - 1 | = 1$, and the value of ρ is $2^{1/2}$. The func-

tion $f'(z)$ is identically zero on C and the value of ρ for $f'(z)$ is infinite. Let the sequence of polynomials $p_n(z)$ converge maximally to $f(z)$ on C. The sequence $p_n'(z)$ cannot converge maximally to $f'(z)$ on C, for if it did, that sequence would converge at every finite point of the plane, uniformly on any closed limited point set. The sequence $p_n(z)$, being convergent at a single point (for instance a point of C), would also converge uniformly on every closed limited point set, and could not represent $f(z)$ on C.

If, however, the complement of C is multiply connected, and if the function $f(z)$ (considered interior to C_ρ as one or more monogenic analytic functions) actually has a singularity on C_ρ, then Theorem 9 and its proof are valid to show that the sequence $p_n'(z)$ converges maximally on C to $f'(z)$. In any case, whether $f(z)$ actually has a singularity on C_ρ or not, the sequence $p_n'(z)$ exhibits the phenomenon of overconvergence.

Maximal convergence may persist even after integration:

THEOREM 10. *If the sequence $p_n(z)$ converges maximally to $f(z)$ on the closed limited point set C which contains more than one point and whose complement is simply connected, then the sequence $P_n(z)$ converges maximally to $F(z)$ on C, where z_0 is a point of C and we introduce the definitions*

$$(17)\qquad P_{n+1}(z) \equiv \int_{z_0}^{z} p_n(z)\,dz, \qquad P_0(z) \equiv 0, \qquad F(z) \equiv \int_{z_0}^{z} f(z)\,dz.$$

If $f(z)$ is analytic interior to C_ρ but has a singularity on C_ρ, the same is true of $F(z)$, when the path of integration in (17) is chosen interior to C_ρ. Let $R < \rho$ be arbitrary and let $\sigma > 1$ be less than ρ/R. For z on or within C_σ we have (Theorem 8)

$$|f(z) - p_n(z)| \leq M/R^n,$$

where M is suitably chosen. If the number L is suitably chosen, there exists a path interior to C_σ from z_0 to an arbitrary point z interior to C_σ whose length (independently of z) is less than L. Then we have for z on or within C_σ,

$$|F(z) - P_{n+1}(z)| = \left| \int_{z_0}^{z} [f(z) - p_n(z)]\,dz \right| \leq ML/R^n, \qquad n \geq 0.$$

In particular this inequality holds for z on C, and it is immaterial whether the inequality holds for the polynomial $P_0(z)$, so the proof is complete.

Iteration of Theorems 9 and 10 obviously yield more general results.

Theorem 10 likewise fails to hold if the complement of C is not simply connected, as we now illustrate. Let C be the point set $|z^2 - 1| \leq 1/4$, and let $f(z)$ in the left-hand oval be identically zero and in the right-hand oval a function analytic interior to but having a singularity on the right-hand oval of the curve $|z^2 - 1| = 1/2$. Let the expansion of $f(z)$ on C be that of §3.4, Theorem 4:

$$(18)\quad f(z) = \sum_{n=0}^{\infty} a_n q_n(z) = a_0 + a_1(z+1) + a_2(z^2-1) + a_3(z+1)(z^2-1) + \cdots;$$

the corresponding sequence of polynomials $p_n(z)$ converges maximally to $f(z)$ on C, and we have $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 2^{1/2}$. We shall integrate (18) term by term from the point $z_0 = -1$ on C to the point $z = 1$ on C. We have

$$\left| \int_{-1}^{1} (z^2 - 1)^n \, dz \right| \geqq \left| 2 \int_{0}^{1} (z - 1)^n \, dz \right| = 2/(n + 1),$$

$$\left| \int_{-1}^{1} (z + 1)(z^2 - 1)^n \, dz \right| \geqq \left| \int_{0}^{1} (z - 1)^n \, dz \right| = 1/(n + 1).$$

Thus we have

$$\overline{\lim_{n\to\infty}} \left| \int_{-1}^{1} q_n(z) \, dz \right|^{1/n} \geqq 1,$$

which implies the divergence of the series

$$\sum_{n=0}^{\infty} a_n \int_{-1}^{1} q_n(z) \, dz,$$

for the n-th term of the series fails to approach zero with $1/n$. It follows that term-by-term integration of the sequence $p_n(z)$ is not valid between any two points which lie interior to different ovals of the locus C or C_ρ.

§4.8. Exact regions of uniform convergence

Under the hypothesis of Theorem 7, a sequence of polynomials $p_n(z)$ of respective degrees n which converges maximally to $f(z)$ on C can clearly converge uniformly in no region containing C_ρ in its interior. Such a sequence can converge at isolated points exterior to C_ρ, as is illustrated by the sequences of §3 5. Nevertheless, Theorem 7 does give an accurate indication of the situation so far as concerns regions of uniform convergence:

THEOREM 11. *If C and $f(z)$ satisfy the hypothesis of Theorem 7, a sequence $p_n(z)$ which converges maximally to $f(z)$ on C converges uniformly in any closed region interior to C_ρ but can converge uniformly in no closed region containing in its interior a point of C_ρ.*

In preparation for the proof of Theorem 11, we prove the following theorem due to Hadamard; this particular proof is due to Pólya and Szegö:

THREE-CIRCLE THEOREM. *Let the function $f(w)$ not identically zero be analytic in the annulus $r_1 \leqq |w| \leqq r_2$ and let M_1 and M_2 be so chosen that we have*

$$|f(w)| \leqq M_1, \quad \text{when } |w| = r_1; \qquad |f(w)| \leqq M_2, \quad \text{when } |w| = r_2.$$

If M denotes

$$\max |f(w)|, \quad \text{when } |w| = r, \ r_1 \leqq r \leqq r_2,$$

then we have

$$M \leqq M_1^{\frac{\log r - \log r_2}{\log r_1 - \log r_2}} \cdot M_2^{\frac{\log r - \log r_1}{\log r_2 - \log r_1}}. \tag{19}$$

The function $w^\alpha f(w)$ is analytic in the given annulus, although not necessarily single-valued, but its modulus is single-valued. Then (Principle of Maximum) $r^\alpha M$ is not greater than the larger of the two quantities $r_1^\alpha M_1$ and $r_2^\alpha M_2$. Those two quantities are equal for the value

$$\alpha = \frac{\log M_1 - \log M_2}{\log r_2 - \log r_1},$$

and the corresponding inequality for $r^\alpha M$ is precisely (19).

More generally, it is sufficient in the hypothesis to assume $f(w)$ analytic interior to the annulus, and

$$\overline{\lim_{|w|\to r_1}} \, |f(w)| \leqq M_1, \qquad \overline{\lim_{|w|\to r_2}} \, |f(w)| \leqq M_2.$$

We shall use the Three-Circle theorem to prove

OSTROWSKI'S THEOREM. *Let the analytic functions $f_k(z)$, $k = 1, 2, \cdots$, converge uniformly to the function $f(z) \not\equiv f_k(z)$ in a simply connected region B, and let B' and B'' be closed subregions interior to B. If m_k' and m_k'' denote the maximum of $|f(z) - f_k(z)|$ in B' and B'' respectively, then there exist two (finite) numbers q and q_1 dependent only on B, B', B'', such that we have for k sufficiently large*

$$0 < q < \frac{\log m_k'}{\log m_k''} < q_1. \tag{20}$$

Map the region B onto the interior of the circle γ: $|w| = 1$ so that an interior point of B' is transformed into the origin. Let the circle γ': $|w| = r_1$ lie interior to the transform of B', and the circle γ'': $|w| = r > r_1$ contain the transform of B'' in its interior. Apply the Three-Circle theorem to the functions $f(z) - f_k(z)$ considered as functions of w, the three circles being γ', γ'', γ, and k being chosen so large that in γ we have $|f(z) - f_k(z)| < 1$. The maximum modulus of $|f(z) - f_k(z)|$ on γ' and γ'' respectively is not greater than m_k' and not less than m_k''. By (19) we have ($M_2 < 1$)

$$m_k'' < (m_k')^{\frac{\log r}{\log r_1}},$$

which implies the first part of (20). The second part of (20) follows by reversing the rôles of B' and B''.

The restriction of the theorem that $f(z) - f_k(z)$ not be identically zero is ordinarily not a serious one. It is clearly a consequence of the theorem that if a sequence converges in B' like a convergent geometric series, then the sequence also converges in B'' like a convergent geometric series. The theorem is readily extended to include the case that B is multiply connected.

We are now in a position to prove Theorem 11. Let B_1 be a closed region which contains a point of C_ρ in its interior, and suppose the sequence $p_n(z)$ converges uniformly in B_1; we shall reach a contradiction. Let B'' bounded by an

analytic Jordan curve be a closed region interior to B_1 also containing a point of C_ρ in its interior. The sequence $p_n(z)$ converges uniformly on and within every C_σ, $\sigma < \rho$, hence converges uniformly in some region B composed of the points of B_1 and the points on and within a contour of some locus C_σ. The present regions B and B'' are to be chosen as the corresponding regions of Ostrowski's theorem. For the closed region B' we may use the closed interior of any contour of a $C_{\sigma'}$ interior to B, and the sequence $p_n(z)$ is known to converge in B' like a convergent geometric series. It follows that in B'' also the sequence $p_n(z)$ converges like a convergent geometric series. We may continue to represent the limit of the sequence by $f(z)$:

$$(21) \qquad |f(z) - p_n(z)| \leq M/\theta^n, \qquad z \text{ in } B'', \qquad \theta > 1.$$

Choose τ, $\rho > \tau > \rho/\theta$. Then by Theorem 8 we have for an arbitrary $\theta_1 < \rho/\tau < \theta$,

$$(22) \qquad |f(z) - p_n(z)| \leq M_1/\theta_1^n, \qquad z \text{ on } C_\tau.$$

Modify B'' if necessary by removing points so that B'' is bounded by an analytic Jordan curve exterior to C_τ; inequality (21) persists. Denote by C' the point set composed of the new B'' plus the closed interior of C_τ. The complement of C' is connected and regular. For an arbitrary $\theta_1 < \rho/\tau$, we have by (21) and (22)

$$|f(z) - p_n(z)| \leq M_2/\theta_1^n, \qquad z \text{ on } C'.$$

By Theorem 6 the sequence $p_n(z)$ converges throughout the interior of $C'_{\rho/\tau}$ and represents a function $f(z)$ single-valued and analytic throughout the interior of $C'_{\rho/\tau}$. The point set C_τ is a proper subset of C', so (§4.1) the curve $[C_\tau]_{\rho/\tau} = C_\rho$ lies interior to $C'_{\rho/\tau}$. Then $f(z)$ is single-valued and analytic throughout the interior of some $C_{\rho'}$, $\rho' > \rho$, in contradiction to the definition of ρ.

Our method of proof yields incidentally the following:

If C and $f(z)$ satisfy the hypothesis of Theorem 7, no sequence $p_n(z)$ which converges maximally to $f(z)$ on C can converge like a convergent geometric series in any region or on any Jordan arc (or indeed on any continuum consisting of more than a single point) exterior to C_ρ.

§4.9. Approximation on more general point sets (irregular case)

We have hitherto considered approximation on a closed limited point set C whose complement K is connected *and regular*. We now indicate a few similar results for the case that C is closed and limited but that its complement K although connected is not regular.

Let the closed limited point sets $C^{(n)}$ (compare §1.3, Theorem 3) each consist of a finite number of mutually exclusive Jordan regions, $C^{(n+1)}$ interior to $C^{(n)}$, C interior to every $C^{(n)}$, every point of K exterior to some $C^{(n)}$. Denote by K_n the complement of $C^{(n)}$ and by $G_n(x, y)$ Green's function for K_n with pole at infinity. At every point (x, y) of K, the functions $G_n(x, y)$ increase monotoni-

cally; compare §4.1. It follows from a well known theorem due to Harnack that the harmonic functions $G_n(x, y)$ either become infinite at every point of K or converge uniformly on every closed limited point set interior to K to a function $G(x, y)$, necessarily harmonic interior to K except at infinity. The function $G(x, y)$, if existent, is expressible in the neighborhood of infinity as $\log r + g(x, y)$, where $g(x, y)$ is harmonic at infinity and $r = (x^2 + y^2)^{1/2}$. Indeed, in some closed neighborhood N_∞ interior to K of the point at infinity we have for n sufficiently large $G_n(x, y) = \log r + g_n(x, y)$, where $g_n(x, y)$ is harmonic in N_∞; the functions $G_n(x, y)$ converge uniformly on the boundary of N_∞, hence the functions $g_n(x, y)$ likewise converge uniformly on that boundary, hence uniformly in the closed neighborhood N_∞ and the limit function $g(x, y)$ is harmonic interior to N_∞. The function $G(x, y)$ is called the *generalized* Green's function for K with pole at infinity.

The function $G(x, y)$ is positive at every (interior) point P of K, for at every such point the sequence $G_n(x, y)$ is defined and positive for suitably large n, and the sequence $G_n(x, y)$ increases monotonically.

The function $G(x, y)$ is independent of the particular sequence of point sets $C^{(n)}$ used in defining it, for let us consider any similar sequence $Q^{(n)}$ with the respective complements K_n' whose Green's functions with pole at infinity are the functions $G_n'(x, y)$. There exists some $C^{(m)}$ interior to any given $Q^{(n)}$ so we have (§4.1): $G(x, y) > G_m(x, y) > G_n'(x, y)$ on every closed finite point set in K_n'. On the other hand, there exists some $Q^{(p)}$ interior to any given $C^{(m)}$, so if m is given we have $G_p'(x, y) > G_m(x, y)$ on any closed finite set in K_m. On any given closed limited point set interior to K, we have for an arbitrary positive ϵ

$$G(x, y) - \epsilon < G_m(x, y) < G(x, y),$$

where m is suitably chosen, and thus we have for suitably chosen p

$$G(x, y) - \epsilon < G_m(x, y) < G_p'(x, y) < G(x, y),$$

and this implies the corresponding inequalities where p is replaced by any larger subscript. That is to say, the function $G(x, y)$ is not only the limit in K of the sequence $G_n(x, y)$ but also of the sequence $G_n'(x, y)$. It can be proved similarly that if the $G_n(x, y)$ become infinite in K so also do the $G_n'(x, y)$.

Whenever K is regular, the functions $G_n(x, y)$ approach uniformly on any closed limited point set in K the Green's function $G(x, y)$ (in the usual sense of the term, §4.1) for K with pole at infinity. For we may choose the sets $C^{(n)}$ as the closed interiors of the loci C_{R_n}, $R_n \to 1$, the R_n naturally being selected so that the C_{R_n} have no multiple points. Then we have $G_n(x, y) = G(x, y) - \log R_n$, which approaches $G(x, y)$ uniformly. This limit $G(x, y)$ is known to be independent of the choice of the point sets $C^{(n)}$.

We return to the case that K is not regular. It follows from the reasoning used above that the loci $C_R^{(n)}$, where R is fixed, vary monotonically. When the sequence $G_n(x, y)$ becomes infinite at every point of K, the loci $C_R^{(n)}$ approach

the boundary of C. When $G_n(x, y)$ approaches the limit $G(x, y)$ uniformly in every closed region interior to K, the loci $C_R^{(n)}$ may approach boundary points of C as well as curves in K. In any case, we denote by C_R the locus which is the limit of the loci $C_R^{(n)}$ in K. The locus C_R is then a closed set and consists of the locus $G(x, y) = \log R$ (if existent) interior to K together with possible points of C itself. The locus C_R is independent of the particular regions $G_n(x, y)$ chosen. The approach of $C_R^{(n)}$ to C_R is monotonic and hence (§1.3, Theorem 4) uniform.

Each locus $G(x, y) = \log R$ consists of a finite or infinite number of Jordan arcs; every Jordan curve composed wholly of such arcs contains points of C in its interior.

In connection with the function $G(x, y)$ and the point set C_R, it is worth while to refer to the work of Kellogg [1923, 1929] and Myrberg [1933] relative to the nature of the function $G(x, y)$ in the neighborhood of a point of C, for instance if P is an isolated point or if some neighborhood of P contains only a set of points of C of capacity (or transfinite diameter) zero.

The analogue of Theorem 5 is valid even if K is not regular: *If $f(z)$ is single-valued and analytic on and within C_R, there exist polynomials $p_n(z)$ of respective degrees n such that* (10) *is valid for z on C.* By the *interior* of C_R we mean here the totality of points separated from the point at infinity by C_R. For suitable choice of m, the function $f(z)$ is single-valued and analytic on and within $C_R^{(m)}$. Then (Theorem 5) polynomials $p_n(z)$ exist such that (10) is valid for z on $C^{(m)}$, hence for z on C. In particular, we note the following: *Let C be an arbitrary closed limited point set and let $f(z)$ be analytic on C; then there exist $R > 1$ and polynomials $p_n(z)$ of respective degrees n such that* (10) *is valid for z on C.*

Reciprocally, if (10) *is valid for z on C, then the function $f(z)$ is single-valued and analytic for z interior to C_R.* In the case that z is separated from the point at infinity wholly by points of C, the analyticity of $f(z)$ in z follows from §1.10, Theorem 16. In any other case the sequence $G_n(x, y)$ approaches a function $G(x, y)$ not the infinite constant. In the latter case we can still introduce the function $w = \phi(z) = e^{G+iH}$, where H is conjugate to G in K, and this function maps K onto the exterior of γ: $|w| = 1$ so that the points at infinity correspond to each other, but now boundary points of K may be transformed into points exterior to γ. When z interior to K approaches points of C, every limit value of $|\phi(z)|$ is greater than or equal to unity. The discussion of §4.6, including both proof of the Lemma and proof of Theorem 6 is valid without change under the present circumstances, and shows that the given sequence $p_n(z)$ converges uniformly on the part of any locus $G(x, y) = \log R_1 < \log R$ interior to K, and indeed on the entire locus C_{R_1}. A point z_0 of K which is separated from the point at infinity by C_R is also separated from the point at infinity by a suitable C_{R_1}, $R_1 < R$; it is sufficient to choose $R_1 > |\phi(z_0)|$. A point z_0 not in K which is separated from the point at infinity by C_R must be a point of C or separated by C from the point at infinity; it is sufficient for us to consider the former case. The point z_0 is not on C_R so we have $\overline{\lim}_{z \text{ in } K,\, z \to z_0}$

$|\phi(z)| = R_2 < R$. Then z_0 is separated from the point at infinity by the locu C_{R_1}, $R_2 < R_1 < R$, so the proof is complete.

The analogue of Theorem 7 applies now directly under the present condition Moreover, the concept of maximal convergence can be introduced here also, an the fundamental properties of a sequence which converges maximally are vali under these new conditions. The existence of a sequence of polynomials co verging maximally on C to an arbitrary function analytic on C follows as in §5.1

Degree of approximation is of particular interest when the given function $f(z$ is analytic on the point set C such that $G_n(x, y)$ becomes infinite at every poin of K. In that case every C_R coincides with the boundary of C, the function $f(z$ is analytic on and within every C_R, so *there exist polynomials* $p_n(z)$ *such that* (10 *is valid for* z *on* C *for every* R, where M depends on R but not on n or z. Th polynomials $p_n(z)$ of degree n exhibited in Theorem 5 depend on R, but (compar §§4.5 and 4.7) the Tchebycheff polynomials (§5.1) for approximation to $f(z)$ o C have the property mentioned for every R and these polynomials do not depen on R. *A necessary and sufficient condition that this property* (i.e. that (10) b valid on the given set C for *every* R for suitably chosen polynomials $p_n(z)$, o in other words that $\lim_{n\to\infty} \mu_n^{1/n} = 0$, $\mu_n = \max [\,|f(z) - p_n(z)|, z$ on C *should hold for every* $f(z)$ *analytic on* C *is that the* $G_n(x, y)$ *should become infinite* *every point of* K, and a necessary and sufficient condition for the latter (Myrber loc. cit.) is that C be of capacity or transfinite diameter zero.

It is readily proved from §3.5, Theorem 6 that the property just mentione holds for an arbitrary closed limited point set C which has only a finite numb of limit points.

The remarks made relative to degree of convergence are of significance in co nection with the problem of interpolation in prescribed points exterior to a give point set. Let C denote the given set plus the exterior points of interpolatio and let $f(z)$ be the given function (assumed analytic) on the given set and t interpolated values in the exterior points. By the present methods one ca study convergence to $f(z)$ on C, and the remaining gap to the study of interpol tion is easy to fill (§§11.1 and 11.2).

In the sequel we shall ordinarily restrict ourselves to the study of closed limite point sets C whose complements K are connected and regular. Many result particularly on the convergence of Tchebycheff polynomials, are stated only i the restricted case, but hold equally well for the more general case.

CHAPTER V

BEST APPROXIMATION BY POLYNOMIALS

§5.1. Tchebycheff approximation

The polynomials of approximation to a given function $f(z)$ considered in Chapters I and II are not uniquely determined and are indeed largely arbitrary. Unique polynomials with extremal properties nevertheless exist and have important properties, as we shall indicate in the present chapter.

Let C be a closed limited point set and let the function $f(z)$ be continuous on C. *The Tchebycheff polynomial $\pi_n(z)$ of degree n for approximation to $f(z)$ on C* is the polynomial $\pi_n(z)$ of degree n such that

$$(1) \qquad \mu_n = \max [\,|f(z) - \pi_n(z)|, \ z \text{ on } C]$$

is not greater than the corresponding expression when $\pi_n(z)$ is replaced by any other polynomial of degree n. We shall prove later (§§12.3 and 12.7) the existence of the polynomial $\pi_n(z)$, and the uniqueness provided C contains at least $n + 1$ points. Such polynomials were first introduced by Tchebycheff, in the case of approximation to a real function on an interval of the axis of reals.

THEOREM 1. *If C is a closed limited point set consisting of infinitely many points, and if the function $f(z)$ can be uniformly approximated by a polynomial on C, then the sequence $\pi_n(z)$ of Tchebycheff polynomials of respective degrees n for approximation to $f(z)$ on C converges uniformly to $f(z)$ on C.*

The polynomial $\pi_n(z)$ is a polynomial not only of degree n but also of degree $n + 1$, so it follows from the definition of μ_n that we have $0 \leqq \mu_{n+1} \leqq \mu_n$. Let us set $\lim_{n\to\infty} \mu_n = \mu \geqq 0$; the limit necessarily exists. It remains to prove $\mu = 0$. Let us assume $\mu > 0$; we shall reach a contradiction. There exists by hypothesis a polynomial $p_m(z)$ of some degree m such that we have

$$|f(z) - p_m(z)| \leqq \mu/2, \qquad z \text{ on } C.$$

Consequently μ_m is not greater than $\mu/2$, and the limit of the μ_n is not greater than $\mu/2$, which is a contradiction.

Theorem 1 clearly has applications to the various situations treated in Chapters I and II. In any case in which the hypothesis of Theorem 1 is satisfied, it might be said that the sequence $\pi_n(z)$ converges to $f(z)$ on C more rapidly than any other sequence of polynomials of respective degrees n.

The concept of Tchebycheff polynomial can be somewhat generalized by the introduction of a weight or norm function $n(z)$, which for the present we take as positive and continuous on C. The Tchebycheff polynomial $\pi_n(z)$ of degree n

for approximation to $f(z)$ on C with norm function $n(z)$ is the polynomial $\pi_n(z)$ of degree n such that

$$(2) \qquad \mu_n = \max [n(z) \mid f(z) - \pi_n(z) \mid, z \text{ on } C]$$

is not greater than the corresponding expression when $\pi_n(z)$ is replaced by any other polynomial of degree n. If $f(z)$ and $n(z)$ are continuous on the closed limited point set C consisting of an infinity of points, such a polynomial $\pi_n(z)$ exists and is unique (§§12.3 and 12.7). The case $n(z) \equiv 1$ is the case already mentioned.

Theorem 1 easily extends to the case of general positive continuous norm function $n(z)$. If we use the notation (2), we have as before $0 \leqq \mu_{n+1} \leqq \mu_n$, so the quantity $\lim_{n\to\infty} \mu_n = \mu \geqq 0$ exists. We prove $\mu = 0$ by assuming the contrary. If we have for z on C the inequality $0 < N_1 \leqq n(z) \leqq N_2$, there exists a polynomial $p_m(z)$ of some degree m such that we have

$$| f(z) - p_m(z) | \leqq \mu/(2N_2), \quad n(z) | f(z) - p_m(z) | \leqq \mu/2, \quad \text{for } z \text{ on } C.$$

Then we have $\mu_m \leqq \mu/2$, an impossibility. The approach to zero (which is now established) of the quantities μ_n implies the approach to zero of the quantities μ_n/N_1, hence the approach to zero of the quantities

$$\max [| f(z) - \pi_n(z) |, z \text{ on } C],$$

and the proof is complete.

A more specific statement than Theorem 1 (either in its original or in its generalized form) can be made in the situation of §4.7:

THEOREM 2. *Let the function $f(z)$ be single-valued and analytic on the closed limited point set C whose complement is connected and regular. Then the sequence of Tchebycheff polynomials $\pi_n(z)$ of respective degrees n of best approximation to $f(z)$ on C converges maximally to $f(z)$ on C. Hence the sequence $\pi_n(z)$ converges to $f(z)$ interior to C_ρ, uniformly on any closed set interior to C_ρ, where ρ is the largest number finite or infinite such that $f(z)$ can be analytically extended from C along paths interior to C_ρ so as to be single-valued and analytic throughout the interior of C_ρ.*

Let $R < \rho$ be arbitrary. There exist (§4.5, Theorem 5) polynomials $p_n(z)$ depending perhaps on R of respective degrees n such that we have

$$(3) \qquad | f(z) - p_n(z) | \leqq M/R^n, \qquad z \text{ on } C.$$

For the Tchebycheff polynomials $\pi_n(z)$ of corresponding degrees we have a fortiori

$$(4) \qquad | f(z) - \pi_n(z) | \leqq M/R^n, \qquad z \text{ on } C,$$

and the proof is complete.

We have used in the proof of Theorem 2 only polynomials $p_n(z)$ *which may depend on R* and which satisfy (3), so we have given a new proof (compare §§4.5 and 4.7) of the existence of a sequence of polynomials converging maximally to

$f(z)$ on C, when $f(z)$ is analytic on the closed limited point set C whose complement is connected and regular. Indeed, the present proof holds (§4.9) also whenever C is closed and limited and its complement K is connected, whether or not K is regular.

Theorem 2 is due in somewhat less general form and without the concept of maximal convergence, to S. Bernstein [1912] for the case that C is a segment of the axis of reals, and to Faber [1920] for the case that C is a Jordan region bounded by an analytic curve.

Theorem 2 easily extends to the case that the polynomials $\pi_n(z)$ are the Tchebycheff polynomials for approximation to $f(z)$ on C with the positive continuous norm function $n(z)$. Let us suppose $0 < N_1 \leqq n(z) \leqq N_2$. Under the new conditions, inequalities (3) and (4) may be replaced by the inequalities for z on C

$$n(z)\,|f(z) - p_n(z)| \leqq M_1/R^n, \qquad n(z)\,|f(z) - \pi_n(z)| \leqq M_1/R^n,$$

and the last inequality yields the desired conclusion:

$$|f(z) - \pi_n(z)| \leqq M_1/(N_1R^n), \qquad z \text{ on } C. \tag{5}$$

In the study of approximation in the sense of Tchebycheff with a norm function $n(z)$ to a function $f(z)$ analytic on and within a closed region C, it is clearly immaterial (Principle of Maximum) whether best approximation is considered on C or on the boundary of C, provided $n(z)$ is either unity or more generally the modulus of a function analytic interior to C, continuous in the corresponding closed region.

The Tchebycheff polynomial is a polynomial of *best approximation*, in the sense that the expression (1) is a minimum for the Tchebycheff polynomial. Approximation of a polynomial $p_n(z)$ to a function $f(z)$ can also be measured by various other expressions, particularly by the use of integrals of the error $|f(z) - p_n(z)|^p$, $p > 0$. The corresponding polynomials of best approximation have various advantages over the Tchebycheff polynomials, notably ease of computation when $p = 2$, so it is appropriate to study the convergence of such polynomials. The new polynomials of best approximation which we shall study exist in every case, and are unique in the case $p > 1$, as we prove in Chapter XII. The phraseology "the polynomial of best approximation" often used in the sequel is to be understood in the sense "any polynomial of best approximation" if such a polynomial fails to be unique. For the sake of generality we permit the use of norm functions in the new measures of approximation. For the present the norm functions are taken positive and continuous where defined, but those restrictions will later (§5.7) be somewhat lightened.

§5.2. Approximation measured by a line integral

THEOREM 3. *Let the function $f(z)$ be single-valued and analytic on the point set C which consists of a closed limited Jordan region bounded by a rectifiable curve Γ. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least p-th powers as measured by the integral*

$$(6) \qquad \int_\Gamma n(z)\,|f(z) - \pi_n(z)|^p\,|dz|, \qquad p > 0,$$

where $n(z)$ is positive and continuous on Γ. Then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.

In the proof of Theorem 3 we shall find it convenient first to prove the

LEMMA. *If Γ is a rectifiable Jordan curve, if $P(z)$ is a polynomial of degree n, and if we have*

$$\int_\Gamma |P(z)|^p\,|dz| \leqq L^p, \qquad p > 0,$$

then we have

$$|P(z)| \leqq LL'R^n, \qquad z \text{ on } \Gamma_R,$$

*where L' depends on Γ, R, and p but not on $P(z)$ nor n.**

Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be the zeros of $P(z)$ exterior to Γ, and let the function $w = \phi(z)$ map the exterior of Γ onto the exterior of γ: $|w| = 1$ so that the points at infinity correspond to each other. The function $\phi(z)$, if suitably defined, is continuous at every point of Γ. Let $m \leqq n$ be the order of the pole of $P(z)$ at infinity, that is, the smallest degree of the polynomial $P(z)$. The function

$$(7) \qquad Q(z) = \frac{P(z)}{[\phi(z)]^m} \cdot \frac{[1 - \overline{\phi(\alpha_1)}\,\phi(z)]\,[1 - \overline{\phi(\alpha_2)}\,\phi(z)] \cdots [1 - \overline{\phi(\alpha_k)}\,\phi(z)]}{[\phi(z) - \phi(\alpha_1)]\,[\phi(z) - \phi(\alpha_2)] \cdots [\phi(z) - \phi(\alpha_k)]}$$

is single-valued and analytic† and different from zero in the extended plane exterior to Γ; the function $\zeta = (1 - \bar\beta w)/(w - \beta)$, $|\beta| > 1$, maps $|w| > 1$ onto $|\zeta| > 1$ and maps $|w| < 1$ onto $|\zeta| < 1$.‡ The function $Q(z)$ is continuous on Γ if appropriately defined on Γ, and on Γ we have $|P(z)| = |Q(z)|$, whence

$$(8) \qquad \int_\Gamma |Q(z)|^p\,|dz| \leqq L^p.$$

The function $[Q(z)]^p/\phi(z)$ is analytic in the extended plane exterior to Γ, zero at infinity, and continuous on Γ, hence is represented by Cauchy's integral

$$\frac{[Q(z)]^p}{\phi(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{[Q(t)]^p\,dt}{\phi(t)(t - z)}, \qquad z \text{ exterior to } \Gamma.$$

It follows now from (8) that for z on Γ_R: $|\phi(z)| = R$ we have $|Q(z)| \leqq LL_1$ where L_1 depends on R but not on n nor on $Q(z)$.

* A suitable number L' can in fact be expressed as the p-th root of a number which depends on Γ but not on p.

† To be sure, this function is not literally defined at infinity or in the points α_j. Here and in the sequel we tacitly assume that the obvious definition is to be supplied in such isolated points.

‡ This function ζ obviously maps $|w| = 1$ onto $|\zeta| = 1$, for $|w| = 1$ implies $\zeta\bar\zeta = (1 - \bar\beta w)(1 - \beta\bar w)/(w - \beta)(\bar w - \bar\beta) = (1 - \bar\beta w)(w - \beta)/(w - \beta)(1 - \bar\beta w) = 1$.

In (7) the modulus of the factor $[\phi(z) - \phi(\alpha_i)]/[1 - \overline{\phi(\alpha_i)}\phi(z)]$ is unity on Γ, hence (Principle of Maximum) is less than unity on Γ_R. Then on Γ_R we have $| P(z) | \leqq | Q(z) | R^m$, from which the lemma follows. The case that $P(z)$ vanishes identically is exceptional for the proof but trivial.

We are now in a position to prove Theorem 3. Let $R < \rho$ be arbitrary and choose R_1, $R < R_1 < \rho$; here and throughout Chapter V we use ρ to denote the number defined in §4.7. There exist (by §4.5, Theorem 5) polynomials $p_n(z)$ of respective degrees n such that we have

$$\int_\Gamma n(z) \, | f(z) - p_n(z) |^p \, | dz | \leqq M/R_1^{np},$$

where M is suitably chosen, and this inequality implies by the definition of the $\pi_n(z)$

$$\int_\Gamma n(z) \, | f(z) - \pi_n(z) |^p \, | dz | \leqq M/R_1^{np}. \tag{9}$$

The well known general inequalities*

$$\begin{aligned} | \chi_1 + \chi_2 |^p &\leqq 2^{p-1} | \chi_1 |^p + 2^{p-1} | \chi_2 |^p, \quad p > 1, \\ | \chi_1 + \chi_2 |^p &\leqq | \chi_1 |^p + | \chi_2 |^p, \quad 0 < p \leqq 1, \end{aligned} \tag{10}$$

yield from (9) by virtue of the boundedness of $1/[n(z)]$

$$\int_\Gamma | \pi_{n+1}(z) - \pi_n(z) |^p \, | dz | \leqq M_1/R_1^{np}.$$

By the Lemma we now find

$$| \pi_{n+1}(z) - \pi_n(z) | \leqq M_2/R^n, \qquad z \text{ on } C_{R_1/R}. \tag{11}$$

This inequality, being valid on $C_{R_1/R}$, is also valid on C. The sequence $\pi_n(z)$ converges uniformly on C to some function $f_1(z)$, and we have by (11) for z on C

$$\begin{aligned} | f_1(z) - \pi_n(z) | \leqq | \pi_{n+1}(z) - \pi_n(z) | + | \pi_{n+2}(z) - \pi_{n+1}(z) | \\ + \cdots \leqq M_2/R^{n-1}(R - 1), \end{aligned} \tag{12}$$

whence

$$\int_\Gamma | f_1(z) - \pi_n(z) |^p \, | dz | \leqq M_3/R^{np},$$

* The first of these inequalities expresses essentially the fact that the curve $y = x^p$, $p > 1$, $x > 0$, is concave upwards; more explicitly, the point $\{(\chi_1 + \chi_2)/2, (\chi_1 + \chi_2)^p/(2^p)\}$ of the curve lies below the line joining the points (χ_1, χ_1^p), (χ_2, χ_2^p) of the curve. The second inequality expresses essentially the fact that the function $y = 1 + x^p - (1 + x)^p$, $x > 0$, $0 < p < 1$ is positive. The proof can be given by showing the positive character of the derivative y'.

where M_3 is suitably chosen. By means of (9) and (10) we may now write

$$\int_\Gamma |f(z) - f_1(z)|^p |dz| \leqq M_4/R^{np}, \qquad n = 0, 1, 2, \cdots,$$

which implies the vanishing of this integral and the identity on Γ of the continuous functions $f(z)$ and $f_1(z)$. Inequality (12) expresses the maximal convergence of the sequence $\pi_n(z)$ to $f(z)$ on C, and Theorem 3 is established.

In the proof of Theorem 3 we have not made use of the fact that the $\pi_n(z)$ are polynomials of best approximation except in the derivation of (9). Our conclusion on the maximal convergence of the $\pi_n(z)$ holds whenever (9) is valid for every $R_1 < \rho$; more generally, we may state the

COROLLARY. *Inequality* (9) *implies inequality* (12) *for z on Γ, where $f_1(z) \equiv f(z)$, provided merely $R < R_1$.*

If the point set C is not a Jordan region as required but is a rectifiable Jordan arc, the conclusion and proof still hold. Nothing is to be changed in the proof except that the function $\phi(z)$ is not now continuous on C, but becomes continuous if the plane is cut along C. More generally, the reasoning holds under still broader conditions. Let us prove

THEOREM 4. *Let the function $f(z)$ be single-valued and analytic on the closed limited point set C which consists of the mutually exterior closed point sets Γ', Γ'', $\cdots$, $\Gamma^{(\nu)}$, where $\Gamma^{(j)}$ is a Jordan region bounded by a rectifiable Jordan curve, or a rectifiable Jordan arc, or a configuration composed of a finite number* of such Jordan regions or arcs or both which does not separate the plane and whose complement is simply connected. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ as measured by the integral* (6) *taken over the boundary Γ of C, where $n(z)$ is positive and continuous on the boundary of C. Then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.*

We prove inequality (9) precisely as before, where $R < \rho$ is arbitrary, $R < R_1 < \rho$:

$$\int_\Gamma n(z) \, |f(z) - \pi_n(z)|^p \, |dz| \leqq M/R_1^{np}.$$

By the Corollary, we have inequality (12) (where $f_1(z)$ and $f(z)$ are identical) for z on each component $\Gamma^{(j)}$, where M_2 depends on j. The corresponding inequality is valid for z on C where M_2 does not depend on j. Theorem 4 follows at once.

Theorem 4 is valid in particular if C is a finite segment of the axis of reals, or a finite number of finite segments of the axis of reals.

Theorem 3 is due in slightly less general form to Szegö [1921] in the case that

* It follows from a theorem due to Seidel [1934] that even an infinite number of arcs or regions may be used here, provided the boundary is rectifiable.

Γ is analytic, $p = 2$ and $n(z) \equiv 1$, and to Smirnoff [1928, 1932] in the case that Γ is rectifiable but with an auxiliary condition, $p = 2$ and $n(z) \equiv 1$. Szegö has also studied [1921b] approximation on the unit circle, $p = 2$, and on a segment of the axis of reals, $p = 2$, with more general weight functions than we consider here. Jackson [1930a] has also studied approximation on a circle, with arbitrary $p > 0$, and has obtained less precise results on overconvergence. Of course approximation on a single segment of the axis of reals has been widely studied in the case $p = 2$; compare §5.7. The situation of Theorem 4 was studied by Lindemann [1881] in the case that the $\Gamma^{(j)}$ are segments of the axis of reals and $\nu = 2$, $p = 2$, $n(z) \equiv 1$ (the polynomials are those of Lamé); by Faber [1922] in the more general situation, $p = 2$, $n(z) \equiv 1$; by Shohat [1933] in the case that the $\Gamma^{(j)}$ are segments of the axis of reals, $p = 2$, without proving overconvergence. In none of these special cases under Theorem 4 was the precise region or regions of overconvergence determined.

§5.3. Approximation measured by a surface integral

THEOREM 5. *Let the function $f(z)$ be single-valued and analytic in a closed limited Jordan region C. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least p-th powers as measured by the integral*

$$\int\!\!\int_C n(z)\, |f(z) - \pi_n(z)|^p \, dS, \qquad p > 0, \tag{13}$$

where $n(z)$ is positive and continuous on C. Then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.

As in the proof of Theorem 3, some preliminary propositions are convenient.

LEMMA I. *Let the function $P(z)$ be analytic on and interior to the circle ω: $|z| = a$. If $p > 0$ is arbitrary, we have*

$$\frac{1}{2\pi a}\int_\omega |P(z)|^p \, |dz| \geqq |P(0)|^p .$$

Let the points $\alpha_1, \alpha_2, \cdots, \alpha_k$ be the zeros of $P(z)$ interior to ω. The function

$$Q(z) = P(z)\, \frac{(a^2 - \bar{\alpha}_1 z)(a^2 - \bar{\alpha}_2 z) \cdots (a^2 - \bar{\alpha}_k z)}{a^k (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_k)} \tag{14}$$

is analytic and different from zero interior to ω, and is continuous on and within ω, as is also the function $[Q(z)]^p$. Then we have Cauchy's integral

$$[Q(0)]^p = \frac{1}{2\pi i}\int_\omega \frac{[Q(t)]^p\, dt}{t}$$

from which follows

$$|Q(0)|^p \leqq \frac{1}{2\pi a}\int_\omega |Q(z)|^p \, |dz| .$$

On ω we have $|P(z)| = |Q(z)|$, and the inequality $|P(z)| \leqq |Q(z)|$ for z interior to ω follows directly from (14) and the Principle of Maximum. The case that $P(z)$ vanishes identically is exceptional for the proof but trivial. Lemma I is established.

LEMMA II. *If the function $P(z)$ is analytic interior to a limited region C, if we have*

$$\int\int_C |P(z)|^p \, dS \leqq L^p, \qquad p > 0, \tag{15}$$

and if C' is an arbitrary closed point set interior to C, then we have

$$|P(z)| \leqq LL', \qquad z \text{ on } C', \tag{16}$$

where L' depends on C' but not on $P(z)$ nor on L.

Let us introduce polar coordinates (r, θ) with origin at an arbitrary point z_0 of C'. By Lemma I we have

$$|P(z_0)|^p \leqq \frac{1}{2\pi} \int_0^{2\pi} |P(z_0 + re^{i\theta})|^p \, d\theta, \tag{17}$$

where the circle $|z - z_0| = r$ and its interior lie interior to C. If we multiply (17) through by $r\,dr$ and integrate from zero to b, we obtain

$$|P(z_0)|^p \leqq \frac{1}{\pi b^2} \int\int_\Omega |P(z)|^p \, dS, \tag{18}$$

where Ω is the region $|z - z_0| \leqq b$, assumed to lie interior to C.

The right-hand member of (18) is not greater than $L^p/(\pi b^2)$, by (15) Then we have established (16), where $L' = (\pi b^2)^{-1/p}$, provided merely b is chosen less than the distance from C' to the boundary of C.

The method of proof follows now to some extent the method used for Theorem 3. Let $R < \rho$ be arbitrary and choose R_1, $R < R_1 < \rho$. There exist (§1.5, Theorem 5) polynomials $p_n(z)$ of respective degrees n such that we have

$$\int\int_C n(z) \, |f(z) - p_n(z)|^p \, dS \leqq M/R_1^{np}.$$

This implies

$$\int\int_C n(z) \, |f(z) - \pi_n(z)|^p \, dS \leqq M/R_1^{np}, \tag{19}$$

from which in turn by the use of Lemma II, we have

$$|f(z) - \pi_n(z)| \leqq M_1/R_1^n, \qquad z \text{ on } C', \tag{20}$$

where C' interior to C is arbitrary.

If we choose C' suitably, C lies interior to the locus $C'_{R_1/R}$ (§2.1, Theorem 2), and for z on $C'_{R_1/R}$ we have (§4.7, Corollary to Theorem 8) from (20),

(21) $$|f(z) - \pi_n(z)| \leqq M_2/R^n.$$

This inequality, being valid for z on $C'_{R_1/R}$ is also valid for z on C, so Theorem 5 is established.

The measure of polynomial approximation used in Theorem 5 was considered by Carleman [1922] in the case $p = 2$, $n(z) \equiv 1$, but without the proof of overconvergence.

In proving Theorem 5 we have used only inequality (19) for the polynomials $\pi_n(z)$. If (19) is valid for every $R_1 < \rho$, then the polynomials $\pi_n(z)$ converge maximally to $f(z)$ on C. More generally, we state the

COROLLARY. *Inequality* (19) *implies inequality* (21) *for z on C, provided merely $R < R_1$.*

Our derivation of (21) for z on C is valid without change if C is no longer a closed Jordan region but is an arbitrary closed limited simply connected region, whether or not the boundary of C separates the plane into more than two regions, and whether or not part of the boundary of C separates other points on the boundary of C from the point at infinity; the point set K the mapping of which defines the curves C_R is now no longer the *complement* of C, but is that one of the regions into which C separates the plane which contains the point at infinity; the region K is simply connected and necessarily regular Moreover, we must assume $f(z)$ analytic not merely in the closed region C, but in every point of the complement of K, or what is the same thing, in every point belonging to C or separated by C from the point at infinity. Then a largest ρ exists such that $f(z)$ is analytic interior to the curve C_ρ. We extend the concept of maximal convergence in the obvious way to include the present situation.

More generally, C may here consist of a finite number of closed limited simply connected regions Γ', Γ'', $\cdots$, $\Gamma^{(\nu)}$, which may or may not have points in common. Inequality (21) for each region $\Gamma^{(j)}$ (where M_2 depends on j) is then proved from (19) by the Corollary, hence inequality (21) holds in its present form on C; under suitable restrictions on $f(z)$, inequality (21) therefore holds at all points separated by C from the point at infinity.

THEOREM 6 *Let C be a closed limited point set consisting of a finite number of closed simply connected regions. Let K (necessarily regular) denote that one of the regions into which the plane is separated by C which contains the point at infinity. Let the function $f(z)$ be analytic at each point of the complement of K. If $\pi_n(z)$ denotes the polynomial of degree n of best approximation to $f(z)$ as measured by the integral* (13), *where $n(z)$ is positive and continuous on C, then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on the complement of K.*

It is not essential here even to suppose the given regions simply connected, provided they are of finite connectivity, for a region C' used in the proof of (21) and interior to a region $\Gamma^{(j)}$ may contain in its interior points not belonging to the set C.

§5.4. Approximation measured by a line integral after conformal mapping of complement

For certain interesting point sets C, such as non-rectifiable Jordan arcs, the convenient measures of approximation (6) and (13) may not apply, so we introduce a new measure of approximation; the new measure of approximation can be used for very general point sets; if the boundary of C is sufficiently smooth, this new measure of approximation is identical with (6) provided the norm functions involved are suitably related to each other.

THEOREM 7. *Let C be a closed limited point set (not a single point) whose complement is simply connected, and let the function $f(z)$ be analytic on C. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C as measured by the integral*

$$\int_\gamma n(w)\,|f(z) - \pi_n(z)|^p\,|dw|, \qquad p > 0, \tag{22}$$

where $n(w)$ is positive and continuous on γ, and where the exterior of C is mapped onto the exterior of γ: $|w| = 1$ so that the points at infinity correspond to each other. Then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.

To be sure, the functions $f(z)$ and $\pi_n(z)$ may not actually be defined on γ, but they are analytic exterior to γ and limited exterior to γ in the neighborhood of γ. The values used in (22) are to be the boundary values [Fatou, 1906] assumed almost everywhere on γ by normal approach to γ, and are integrable on γ in the sense of Lebesgue. Denote the mapping function by $w = \phi(z)$, $z = \psi(w)$.

LEMMA. *Let the function $P(w)$ be analytic exterior to γ: $|w| = 1$ except for a pole at infinity of order not greater than n, have at most a finite number of zeros exterior to γ,* and be uniformly bounded exterior to γ in the neighborhood of γ. Then the inequality*

$$\int_\gamma |P(w)|^p\,|dw| \leqq L^p, \qquad p > 0,$$

implies

$$|P(w)| \leqq LL'R^n, \qquad |w| = R,$$

where L' does not depend on $P(w)$ or n.

The proof of this lemma is precisely the same as the proof of the Lemma of §5.2. We proceed to establish Theorem 7.

Let $R < \rho$ be arbitrary, and choose R_1 and σ_1, $R < R_1 < \rho$, $1 < \sigma_1 < \rho/R_1$. There exist (§4.7, Theorem 8) polynomials $p_n(z)$ of respective degrees n such that we have uniformly for all n and for all $\sigma \leqq \sigma_1 < \rho/R_1$, $\sigma > 1$,

* This condition is unnecessary for the truth of the Lemma, but is a convenience in the proof. If this condition is not made, the use of the Blaschke product (§10.1) is natural.

$$\int_{|w|=\sigma} n(w/\sigma)\,|f(z) - p_n(z)|^p\,|dw| \leqq M/R_1^{np},$$

where M is suitably chosen; by allowing σ to approach unity (the integrand is uniformly bounded and approaches a limit) we obtain

$$\int_\gamma n(w)\,|f(z) - p_n(z)|^p\,|dw| \leqq M/R_1^{np}.$$

This inequality implies by the definition of the polynomials $\pi_n(z)$

$$\int_\gamma n(w)\,|f(z) - \pi_n(z)|^p\,|dw| \leqq M/R_1^{np}, \tag{23}$$

which by the use of (10) yields

$$\int_\gamma |\pi_{n+1}(z) - \pi_n(z)|^p\,|dw| \leqq M_1/R_1^{np}.$$

The conditions of the Lemma are now fulfilled except in the trivial case that $\pi_{n+1}(z)$ and $\pi_n(z)$ are identically equal, so we have

$$|\pi_{n+1}(z) - \pi_n(z)| \leqq M_2/R^{n+1}, \qquad z \text{ on } C_{R_1/R}. \tag{24}$$

This inequality, valid on $C_{R_1/R}$, is also valid on C. Hence the sequence $\pi_n(z)$ converges uniformly on C to some function $f_1(z)$ analytic on C. The identity on C of the functions $f(z)$ and $f_1(z)$ remains to be established.

Overconvergence of the sequence $\pi_n(z)$ takes place, so the function $f_1(z)$ is analytic in some neighborhood of C exterior to C. For the boundary values on γ of the two functions $f(z)$ and $f_1(z)$ (analytic in some neighborhood of γ exterior to γ) we prove as in §5.2,

$$\int_\gamma |f[\psi(w)] - f_1[\psi(w)]|^p\,|dw| = 0,$$

which implies the vanishing almost everywhere on γ of the function $f[\psi(w)] - f_1[\psi(w)]$. The function $\psi(w)$ approaches a limit as w exterior to γ approaches γ radially, except for a set of values of w of measure zero, the values of w for which the limit exists correspond to points of C accessible from the exterior of C. Almost all values of w on γ are points at which $f[\psi(w)]$ and $f_1[\psi(w)]$ are equal and which correspond to accessible points of C. Any arc of γ necessarily contains points at which $f[\psi(w)]$ and $f_1[\psi(w)]$ are equal and which correspond to accessible points of C. If $z = \alpha$ is an arbitrary point of the boundary of C, any circle C_1 with α as center which cuts C possesses one or more arcs which together with part of the boundary of C bound a simply connected region D exterior to C but interior to C_1. The region D corresponds to a region exterior to γ but bounded in part by one or more arcs of γ. The circle C_1 therefore contains in its interior (an arbitrary neighborhood of α) accessible points of the boundary of C at which $f(z)$ and $f_1(z)$ are equal. Then $f(z)$ and $f_1(z)$ are equal in points everywhere

dense on the boundary of C, and hence are identically equal. The proof is complete.

COROLLARY. *Inequality* (23) *implies inequality* (24) *for z on C, where $R < R_1$; if also the given function $f(z)$ is analytic on C, inequality* (24) *implies*

$$|f(z) - \pi_n(z)| \leq M_3/R^n, \qquad z \text{ on } C.$$

The following is a generalization of Theorem 7:

THEOREM 8. *Let C be a closed limited point set whose complement is connected, which consists of the components $\Gamma', \Gamma'', \cdots, \Gamma^{(\nu)}$, each of which consists of more than a single point. Let the function $f(z)$ be analytic on C and let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C as measured by the sum*

$$\sum_{k=1}^{\nu} \int_{\gamma_k} n_k(w) \, |f(z) - \pi_n(z)|^p \, |dw|, \qquad p > 0,$$

where $n_k(w)$ is positive and continuous on γ_k and where the exterior of $\Gamma^{(k)}$ is mapped onto the exterior of γ_k: $|w| = 1$ so that the points at infinity correspond to each other. Then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.

Precisely as in the proof of Theorem 7, we can establish

$$\sum_{k=1}^{\nu} \int_{\gamma_k} n_k(w) \, |f(z) - \pi_n(z)|^p \, |dw| \leq M/R_1^{np},$$

and by the Corollary this implies inequality (24) which now holds (with M_2 depending on j) on each $\Gamma^{(j)}$. Then (24) holds in its present form for z on C, and the remainder of the proof proceeds as before.

§5.5. Approximation measured by a line integral after conformal mapping of interior

Another measure of approximation somewhat similar to (22) is obtained by mapping the *interior* of a given region onto the interior of the unit circle γ of the w-plane. This measure of approximation may apply when the measure of §5.2 cannot be used, and may apply (see §5.8; the measure of §5.4 is not applicable) when the function is not analytic in the given closed region.

THEOREM 9. *Let C be an arbitrary closed limited simply connected region, and let the function $f(z)$ be analytic at every point either on C or separated by C from the point at infinity. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C as measured by the integral*

$$(25) \qquad \int_{\gamma} n(w) \, |f(z) - \pi_n(z)|^p \, |dw|, \qquad p > 0,$$

where $n(w)$ is positive and continuous on γ, and where the interior of C is mapped

onto the interior of γ: $|w| = 1$. *Then the sequence* $\pi_n(z)$ *converges maximally to* $f(z)$ *on* C.

The measure of approximation (25) obviously depends on the particular point z_0 interior to C which is made to correspond to the point $w = 0$, but this measure of approximation with a given choice of z_0 and a given norm function $n(w)$ is equivalent to the corresponding measure of approximation with an arbitrary choice of z_0 and a suitable norm function.

The functions $f(z)$ and $\pi_n(z)$ need not be defined on γ by the conformal map, but they are analytic and uniformly limited interior to γ, so their boundary values exist; the integral (25) is to be understood as involving those boundary values.

LEMMA. *If the function* $P(w)$ *is analytic and uniformly limited interior to* γ: $|w| = 1$, *and has only a finite number of zeros interior to* γ,* *then the inequality*

$$\int_\gamma |P(w)|^p \, |dw| \leqq L^p, \qquad p > 0,$$

implies for z *on an arbitrary set* γ' *interior to* γ

$$|P(w)| \leqq LL', \tag{26}$$

where L' *depends on* γ' *but not on* $P(w)$.

Let the zeros of $P(w)$ interior to γ be $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$. The function

$$Q(w) = P(w) \frac{(1 - \bar{\alpha}_1 w)(1 - \bar{\alpha}_2 w) \cdots (1 - \bar{\alpha}_\lambda w)}{(w - \alpha_1)(w - \alpha_2) \cdots (w - \alpha_\lambda)} \tag{27}$$

is analytic, uniformly limited, and different from zero interior to γ; on γ we have $|Q(w)| = |P(w)|$. Cauchy's integral formula is valid:

$$[Q(w)]^p = \frac{1}{2\pi i} \int_\gamma \frac{[Q(t)]^p \, dt}{t - w}, \qquad w \text{ interior to } \gamma,$$

where the values of $[Q(t)]^p$ used in the integrand are the boundary values taken on by normal approach. We have immediately $|Q(w)| \leqq LL'$ for z on γ', and from (27) we have $|P(w)| \leqq |Q(w)|$, z on γ'. The proof of the lemma is complete.

Let $R < \rho$ be given; choose R_1, $R < R_1 < \rho$. There exist polynomials $p_n(z)$ of respective degrees n such that the integral (25) with $\pi_n(z)$ replaced by $p_n(z)$ is not greater than M/R_1^{np}, where M is suitably chosen. Hence (25) in its present form is not greater than M/R_1^{np}. Each function $f(z) - \pi_n(z)$ either vanishes identically or has at most a finite number of zeros interior to C (or γ). By the use of inequalities (10) and the Lemma we have

* This restriction is merely for convenience in proof; compare §10.1 on the Blaschke product.

$$|f(z) - \pi_n(z)| \leqq M_1/R_1^n, \qquad z \text{ on } C',$$

where C' is an arbitrary point set interior to C. If C' is suitably chosen, the locus $C'_{R_1/R}$ lies exterior to C (§2.1, Theorem 2), so (§4.7, Corollary to Theorem 8) for z on $C'_{R_1/R}$, and therefore for z on C, we have

$$|f(z) - \pi_n(z)| \leqq M_2/R^n.$$

The proof is complete.

The following theorem can now be proved by the method already used several times:

THEOREM 10. *Let C be an arbitrary closed limited point set consisting of the closed simply connected regions $\Gamma', \Gamma'', \cdots, \Gamma^{(\nu)}$, and let K denote that one of the regions into which the plane is separated by C which contains the point at infinity. Let the function $f(z)$ be analytic at every point of the complement of K, and let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on C as measured by the sum*

$$\sum_{k=1}^{\nu} \int_{\gamma_k} n_k(w)\, |f(z) - \pi_n(z)|^p\, |dw|, \qquad p > 0,$$

where $n_k(w)$ is positive and continuous on γ_k and where the interior of $\Gamma^{(k)}$ is mapped onto the interior of γ_k: $|w| = 1$. Then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.

Results similar to Theorems 9 and 10 exist, where a simply connected region C or $\Gamma^{(j)}$ is mapped onto the interior of γ: $|w| = 1$, and where approximation is measured by a surface integral taken over the interior of γ. There is now no difficulty in proving these new results; the treatment is left to the reader.

Theorems 2 (complement of C simply connected), 3, 5, 7, 9 are due to Walsh [1930a, 1931]; Theorem 2 (general case), and Theorems 4, 6, 10, in a less general form, are due to Walsh and Russell [1934]; the present formulations are due to Walsh [1933e].

§5.6. Point sets with infinitely many components

The Tchebycheff measure of approximation (§5.1) is applicable whether the point set involved has a finite or infinite number of components, but our other measures of approximation (§§5.2–5.5) can in their present forms be applied only in the case of a finite number of components. In certain cases, however, our methods can be extended to apply to the more general case, as we proceed to illustrate by a single relatively simple example; for definiteness and for the sake of generality in the kind of point sets allowed we choose the measure of approximation considered in §5.4.

THEOREM 11. *Let C be a closed limited point set whose complement is connected and regular, and let C consist of a denumerably infinite set of components $\Gamma', \Gamma'', \cdots$,*

Let C have the property that given $R > 1$, there exists ν depending on R such that $(\Gamma' + \Gamma'' + \cdots + \Gamma^{(\nu)})_R$ contains C in its interior. Let approximation of a polynomial $p_n(z)$ of degree n to a function $f(z)$ analytic on C be measured on $\Gamma^{(k)}$ by the integral

$$\epsilon_k^{(n)} = \int_{\gamma_k} n_k(w) \, | f(z) - p_n(z) |^p \, | dw |, \qquad p > 0,$$

where the exterior of $\Gamma^{(k)}$ is mapped onto the exterior of γ_k: $| w | = 1$ so that the points at infinity correspond to each other and where $n_k(w)$ is positive and continuous and not greater than unity on γ_k; any particular $\Gamma^{(k)}$ which consists of merely a single point is simply to be omitted in considering the $\epsilon_k^{(n)}$. We introduce as measure of approximation of $p_n(z)$ to $f(z)$ on C the quantity $\epsilon^{(n)} = \sum \epsilon_k^{(n)}/2^k$. If $f(z)$ is an arbitrary function analytic on C and if $\pi_n(z)$ is the polynomial of degree n of best approximation to $f(z)$ on C, then the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C.

By the method already used a number of times, we have for an arbitrary $R < \rho$, $R < R_1 < R_2 < \rho$,

$$\epsilon_k^{(n)} \leqq M/R_2^n, \qquad \epsilon^{(n)} \leqq M/R_2^n,$$

for the measure of approximation of $\pi_n(z)$ to $f(z)$. Then we have also

$$\epsilon_k^{(n)} \leqq 2^k \epsilon^{(n)} \leqq 2^k M/R_2^n,$$

so (by the Corollary to Theorem 7) on each component $\Gamma^{(k)}$ we have

$$| f(z) - \pi_n(z) | \leqq M^{(k)}/R_1^n.$$

By §4.7, Corollary to Theorem 8, we have on the entire set $[\Gamma' + \Gamma'' + \cdots + \Gamma^{(\nu)}]_{R_1/R}$, and therefore by suitable choice of ν for z on C

$$| f(z) - \pi_n(z) | \leqq M_1/R^n,$$

where M_1 is suitably chosen. The proof is complete.

It is not essential that all the components of C be included in the sequence $\Gamma^{(k)}$ provided the property already mentioned persists, nor is it even essential that the $\Gamma^{(k)}$ be *components* of C.

Various point sets which are of interest satisfy the requirements of Theorem 11; for instance we may choose Γ' as the segment $-1 \leqq x \leqq 1$, $y = 0$, and $\Gamma^{(k)}$ $(k > 1)$ as the segment $-1 \leqq x \leqq 1$, $y = 1/k$. The curve $\Gamma_R^{(k)}(k > 1)$ is the ellipse whose foci are the points $(-1, 1/k)$ and $(1, 1/k)$ and whose semimajor axis is $\frac{1}{2}(R + 1/R)$. The curve $\Gamma_R^{(k)}$ is interior to the locus $(\Gamma' + \Gamma'' + \cdots + \Gamma^{(\nu)})_R$, $\nu > k$, and hence if R is given the number ν can be determined so that the latter locus contains C in its interior.

The method we have used in the proof of Theorem 11 is in reality the iteration of the method previously used in the proof of Theorem 5 and elsewhere. The iteration can naturally be repeated any finite number of times, and a corre-

spondingly more general theorem can be stated. Compare in this connection Theorem 14 and its method of proof.

One interesting application of Theorem 11 and the method of its proof is the study of approximation where the given norm function is not assumed different from zero; the point sets $\Gamma^{(k)}$ belonging to C may still exist such that the reasoning can be applied. The significance of this remark appears in the following section.

§5.7. Generality of weight functions

In §§5.2–5.6 we have supposed for simplicity the norm function $n(z)$ or $n(w)$ positive and continuous on the point set on which it is defined. For some purposes it is desirable to lighten this restriction, particularly by admitting points where $n(z)$ or $n(w)$ vanishes; we need merely remind the reader of the large variety of important polynomials orthogonal on a segment of the axis of reals with respect to a weight function which may vanish at various points; these polynomials are intimately related to the polynomials $\pi_n(z)$ already studied, as we shall establish later (Chapter VI). We now indicate briefly how our previous requirements can be modified.

Weight functions have entered into our reasoning at two points: first, the inequality

$$|f(z) - p_n(z)| \leqq M/R^n, \qquad z \text{ on } C,$$

is used to prove the inequality

$$\int n(z)\, |f(z) - p_n(z)|^p\, |dz| \leqq M_1/R^{np}$$

or a similar inequality for some other integral taken over C or its boundary or over γ. It is clearly sufficient here if the function $n(z)$ [or $n(w)$] is integrable in the sense of Lebesgue, a restriction which is relatively moderate.

The weight function also enters our reasoning in deriving an inequality such as

$$\int |f(z) - \pi_n(z)|^p\, |dz| \leqq M_3/R^{np} \tag{28}$$

from the inequality

$$\int n(z)\, |f(z) - \pi_n(z)|^p\, |dz| \leqq M_2/R^{np}. \tag{29}$$

A sufficient condition for this conclusion is clearly that $1/n(z)$ should be bounded, or that $n(z)$ should be bounded from zero.

THEOREM 12. *In Theorems* 3–11 *it is sufficient if the non-negative norm function $n(z)$ is integrable and bounded from zero.*

Still another method of concluding (28) from (29) is by means of the Hölder inequality (which for $\alpha = 1/2$ becomes the Schwarz inequality)

$$(30) \qquad \left| \int F^\alpha G^{1-\alpha} \right| \leqq \left(\int | F | \right)^\alpha \left(\int | G | \right)^{1-\alpha}, \qquad 0 < \alpha < 1,$$

whose validity requires merely the existence of the integrals on the right. Under the present hypothesis of (29) and if $[n(z)]^{-\beta}$, $\beta > 0$, is integrable, we can set $\alpha = 1/(1 + \beta)$:

$$(31) \quad \int | f(z) - \pi_n(z) |^{p(1-\alpha)} | dz | \leqq \left(\int \frac{| dz |}{[n(z)]^{(1-\alpha)/\alpha}} \right)^\alpha \left(\int n(z) | f(z) - \pi_n(z) |^p | dz | \right)^{1-\alpha}$$
$$\leqq M_4 / R^{np(1-\alpha)}.$$

To be sure, inequality (31) is not precisely the same form as (28), for (31) involves the $p(1 - \alpha)$-th power, not the p-th power of $| f(z) - \pi_n(z) |$ Nevertheless, the reasoning we have previously used (§§5.2–5.6) is effective when applied to (31) instead of (28).

THEOREM 13. *In Theorems* 3–11 *it is sufficient if the non-negative norm function* $n(z)$ *is integrable and if some negative power of* $n(z)$ *is integrable.*

The condition that some negative power of $n(z)$ be integrable has been used by Jackson [1933] in the study of approximation in the real domain. Theorems 12 and 13 apply to all the cases previously studied in which an integral measure of approximation is used. Still further results can be obtained in some of those cases. For instance, in Theorem 5 (a similar remark applies to Theorem 6) it is sufficient if for every $\delta > 0$ the function $n(z)$ is non-negative, integrable in C, and is bounded from zero (or more generally if some negative power of $n(z)$ is integrable) in some Jordan region interior to C which contains all points interior to C at a distance from the boundary of C not greater than δ. We proceed to discuss in more detail the situations of Theorems 3 and 4

THEOREM 14. *Let C be an arbitrary rectifiable Jordan arc, let the non-negative function $n(z)$ be integrable on C, and on every closed Jordan subarc of C containing no point of a certain closed reducible set S on C let some negative power (depending on the subarc) of $n(z)$ be integrable. Then the sequence of polynomials $\pi_n(z)$ of degree n of best approximation on C to the function $f(z)$ analytic on C as measured by the integral*

$$\int_C n(z) \, | f(z) - \pi_n(z) |^p \, | dz |, \qquad p > 0,$$

converges maximally to $f(z)$ on C.

The reasoning to be used in proving Theorem 14 is well illustrated by a special case. Let C be a closed limited interval of the axis of reals, and let the function

$n(z)$ be continuous on C, positive at interior points of the interval, zero at both ends of the interval. Let $R < \rho$ be arbitrary and choose R_1 and R_2, $R < R_1 < R_2 < \rho$. There exist polynomials $p_n(z)$ of respective degrees n such that we have

$$\int_C n(z) \mid f(z) - p_n(z) \mid^p \mid dz \mid \leqq M/R_2^{np},$$

so the corresponding inequality holds for the polynomials $\pi_n(z)$ of best approximation. If C' is a variable subinterval of C, we have

$$\int_{C'} n(z) \mid f(z) - \pi_n(z) \mid^p \mid dz \mid \leqq M/R_2^{np}.$$

By the Corollary to Theorem 3 (applied to a Jordan arc rather than a Jordan curve) we have for z on C'

$$\mid f(z) - \pi_n(z) \mid \leqq M_1/R_1^n.$$

Then for z on $C'_{R_1/R}$ we have (§4.7, Corollary to Theorem 8)

$$\mid f(z) - \pi_n(z) \mid \leqq M_2/R^n.$$

The curve $C'_{R_1/R}$ is an ellipse whose foci are the end-points of C' and whose eccentricity depends only on R_1/R, so if C' is suitably chosen the interval C lies interior to $C'_{R_1/R}$. The inequality valid for z on $C'_{R_1/R}$ is also valid for z on C, so the sequence $\pi_n(z)$ converges maximally to $f(z)$ on C, as we were to prove. We turn now to the proof of Theorem 14 in its general form.

Let $R < \rho$ be arbitrary and let C' be any closed set consisting of a finite number of subarcs of C but containing only a finite number of points of S. We shall prove the inequality

$$(32) \qquad \mid f(z) - \pi_n(z) \mid \leqq M/R^n, \qquad z \text{ on } C'.$$

Let Γ be a variable closed subset of C' which consists of a finite number of arcs belonging to C and contains no point of S. Choose R_1 and R_2, $R < R_1 < R_2 < \rho$. The inequality

$$\int_\Gamma n(z) \mid f(z) - \pi_n(z) \mid^p \mid dz \mid \leqq M_1/R_2^{np}$$

is valid, and by virtue of the method used in Theorem 13 and the Corollary to Theorem 3 we have

$$\mid f(z) - \pi_n(z) \mid \leqq M_2/R_1^n, \qquad z \text{ on } \Gamma.$$

Let Γ be chosen as C' with neighborhoods of the points of S belonging to C' deleted. If these neighborhoods are chosen sufficiently small, the locus $\Gamma_{R_1/R}$ contains the point set C' in its interior (§2.1, Corollary to Theorem 2), from which we conclude (§4.7, Corollary to Theorem 8) inequality (32) for z on $\Gamma_{R_1/R}$ and hence for z on C' as indicated.

By use of the result just proved valid on a variable point set C', and by a new application of the method used (i.e. involving the results of §2.1), it follows that for an arbitrary $R < \rho$ and for an arbitrary closed set C'' consisting of a finite number of subarcs of C but containing only a finite number of limit points of S, we have

$$|f(z) - \pi_n(z)| \leqq M'/R^n, \qquad z \text{ on } C''.$$

The entire theorem [Walsh, 1934d] is similarly proved by renewed application of the method given. The formal proof may be given by induction, and is now readily supplied by the reader.

Precisely the same method may be used to prove

THEOREM 15. *In Theorem 4 it is sufficient if the non-negative function $n(z)$ is integrable on Γ, and if on every closed Jordan subarc belonging to Γ and containing no point of a certain closed reducible set S on Γ some negative power (depending on the subarc) of $n(z)$ is integrable.*

Even Theorem 15 may be extended, in the spirit of Theorem 11.

Theorem 15 is of particular interest when C is a subset of the axis of reals. The corresponding situation for $p = 2$ with various other restrictions on the weight function has been studied by many writers, for instance C. Neumann [1862], Szegö [1921b], Faber [1922], S. Bernstein [1930], Shohat [1933], and others. In case C has more than one component, the present results are more specific than any previously published results concerning the regions of convergence of the sequence of polynomials of best approximation; for the case that C is a single line segment the weight functions admitted by Szego and Bernstein are more general in some respects and less general in other respects than those admitted in Theorems 14 and 15. Faber has also proved some results ($p = 2$) on approximation on several line segments or in several Jordan regions analogous to Theorems 14 and 15, but with conclusions less specific than the present ones.

§5.8. Approximation of functions not analytic on closed set considered

The general study of approximation to arbitrary functions not assumed analytic on the given closed sets, as measured by the methods of approximation that we have used in §§5.2–5.5, is not yet in a completely satisfactory state in the literature. Nevertheless, many of our preceding results and methods are applicable with little or no modification, as we now proceed to indicate.

THEOREM 16. *Let C be an arbitrary rectifiable Jordan curve, and let the functions $\psi(z)$ and $\psi_n(z)$ be represented interior to C by the integrals*

$$(33) \qquad \psi(z) = \frac{1}{2\pi i}\int_C \frac{\Psi(t)\,dt}{t - z}, \qquad \psi_n(z) = \frac{1}{2\pi i}\int_C \frac{\Psi_n(t)\,dt}{t - z};$$

the functions $\Psi(z)$ *and* $\Psi_n(z)$ *need not be boundary values of* $\psi(z)$ *and* $\psi_n(z)$ *respectively.** *Then the equation*

$$\lim_{n\to\infty} \int_C |\Psi(t) - \Psi_n(t)|^p \, |dt| = 0, \qquad p > 1,$$

implies

$$\lim_{n\to\infty} \psi_n(z) = \psi(z) \tag{34}$$

for z *interior to* C, *uniformly on any closed set interior to* C.

In inequality (30) we set

$$F(t) = [\Psi(t) - \Psi_n(t)]^p, \qquad G(t) = (t - z)^{p/(1-p)}, \qquad \alpha = 1/p.$$

By means of (33) we have directly for z interior to C

$$|\psi(z) - \psi_n(z)| \leqq \frac{1}{2\pi}\left(\int_C |\Psi(t) - \Psi_n(t)|^p \, |dt|\right)^{1/p}\left(\int_C \frac{|dt|}{|t - z|^{p/(p-1)}}\right)^{(p-1)/p},$$

from which the theorem follows.

It is clear that the method of proof used for Theorem 16 can also be used in the proof of Theorem 3. This present method of proof is somewhat simpler than the one previously used, but has the disadvantage of applying only for values of p greater than unity, and of requiring the application of §2.1, Theorem 2.

The method of proof of Theorem 16 applies with little modification in the proof of the

COROLLARY. *Let* $C', C'', \cdots, C^{(\nu)}$ *be mutually exterior rectifiable Jordan curves and let the function* $f(z)$ *be analytic interior to each* $C^{(k)}$, *continuous in the corresponding closed region. Let* $\pi_n(z)$ *be the polynomial of degree* n *of best approximation to* $f(z)$ *on the set* C: $C' + C'' + \cdots + C^{(\nu)}$ *in the sense of least* p-*th powers* ($p > 1$) *with a positive continuous norm function, as measured by a line integral over* C. *Then the sequence* $\pi_n(z)$ *approaches* $f(z)$ *interior to* C, *uniformly on any closed set interior to* C.

There exists (by §2.8, Theorem 15) some set of polynomials $p_n(z)$ not necessarily of respective degrees n such that we have

$$\lim_{n\to\infty} \int_C n(z) \, |f(z) - p_n(z)|^p \, |dz| = 0.$$

Then the monotonic non-increasing set of numbers

$$\int_C n(z) \, |f(z) - \pi_n(z)|^p \, |dz| \tag{35}$$

* For example, we may have $\Psi(t) = 1/t$, where the origin is interior to C. It is then readily computed, for instance by separating the integrand into partial fractions, that $\psi(z)$ is identically zero for z interior to C. Compare also §§6.10 and 6.11.

also approaches zero; compare §5.1. The Corollary is now a consequence of Theorem 16.

In the case that the curve C of Theorem 16 or the curves $C^{(k)}$ of the Corollary are sufficiently smooth, certain other results can be established by the use of conformal mapping. Such a measure of approximation as (35) can be written as

$$\sum_{k=1}^{\nu} \int_{\gamma_k} n_k(w) \,|\, f(z) - \pi_n(z) \,|^p \,|\, dw \,|, \tag{36}$$

where the function $w = \phi_k(z)$ maps the interior of $C^{(k)}$ onto the interior of γ_k: $|\, w \,| = 1$, and where we set $n_k(w) = n(z) \,|\, dz/dw \,|$. Under suitable restrictions on the curves $C^{(k)}$, this function $n_k(w)$ is positive and continuous on the γ_k, or more generally satisfies the hypothesis of Theorem 13. The expression (36) is in the precise form for the use of Theorem 18 below and its Corollary.

We proceed with the analogue of Theorem 16 where a surface integral is used as a measure of approximation:

Theorem 17. *Let C be an arbitrary limited region and let the functions $\psi(z)$ and $\psi_n(z)$ be analytic interior to C. Then the equation*

$$\lim_{n\to\infty} \iint_C |\, \psi(z) - \psi_n(z) \,|^p \, dS = 0, \qquad p > 0,$$

implies

$$\lim_{n\to\infty} \psi_n(z) = \psi(z)$$

for z interior to C, uniformly on any closed set interior to C.

This theorem is an immediate consequence of §5.3, Lemma II. The same method of proof yields

Corollary 1. *The equation*

$$\lim_{m,n\to\infty} \iint_C |\, \psi_m(z) - \psi_n(z) \,|^p \, dS = 0, \qquad p > 0,$$

where the functions are analytic interior to the limited region C, implies the uniform convergence of the sequence $\psi_n(z)$ on any closed set interior to C.

Corollary 2. *Let $C', C'', \cdots, C^{(\nu)}$ be mutually exterior Jordan curves, and let the function $f(z)$ be analytic interior to each $C^{(k)}$, its p-th power ($p > 0$) integrable in the corresponding open region. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on the set C (consisting of the interiors of the $C^{(k)}$) in the sense of least p-th powers with a positive continuous norm function as measured by a surface integral over C. Then the sequence $\pi_n(z)$ approaches $f(z)$ interior to C, uniformly on any closed set interior to C.*

The proof is the direct analogue of the proof of the Corollary to Theorem 16, by the use of §2.7, Theorem 14 and the method of §2.8. The hypothesis of Corollary 1 implies (method of §6.3) for $p = 2$ the existence of a function $\psi(z)$ satisfying the hypothesis of Theorem 17.

THEOREM 18. *Let γ be the unit circle $|z| = 1$, let the functions $\psi(z)$ and $\psi_n(z)$ be analytic interior to γ, and let the functions $\psi(z) - \psi_n(z)$ have the property that*

$$\lim_{\substack{r \to 1 \\ r<1}} \int_0^{2\pi} | \psi(re^{i\theta}) - \psi_n(re^{i\theta}) |^p \, d\theta = L_n^p, \qquad p > 0,$$

exists. If the quantities L_n approach zero with $1/n$, then (34) is valid for z interior to γ, uniformly on any closed set interior to γ.

The functions $\psi(z) - \psi_n(z)$ may have an infinite number of zeros interior to γ; nevertheless, the corresponding Blaschke product (see §10.1) converges [F. Riesz, 1923], and the functions $\psi(z) - \psi_n(z)$ have boundary values (found by normal approach) almost everywhere on γ. The proof of the Lemma of §5.5 applies now with only obvious modifications, and Theorem 18 follows at once.

COROLLARY. *Let $C', C'', \cdots, C^{(\nu)}$ be mutually exterior Jordan curves and let the function $f(z)$ be analytic interior to each $C^{(k)}$, continuous in the corresponding closed region. Let $\pi_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on the set C: $C' + C'' + \cdots + C^{(\nu)}$ in the sense of least p-th powers ($p > 0$) with a positive continuous norm function as measured by*

$$\sum_{k=1}^{\nu} \int_{\gamma_k} n_k(w) \, | f(z) - \pi_n(z) |^p \, | dw |,$$

where the interior of $C^{(k)}$ is mapped onto the interior of γ_k: $|w| = 1$. Then the sequence $\pi_n(z)$ approaches $f(z)$ interior to C, uniformly on any closed set interior to C.

The proof follows the proofs previously given.

It is worth remarking that the Lemma of §5.5 and its proof extend to the case that γ is an arbitrary rectifiable Jordan curve instead of a circle. But this extension involves proof of the existence of boundary values on γ and of the validity of the corresponding Cauchy integral [compare Seidel, 1934], and is beyond the scope of the present work. Such extension yields the proof of the Corollary of Theorem 16 with the restriction $p > 0$ instead of $p > 1$.

It is not difficult to establish results analogous to the Corollaries to Theorems 16–18 where a double integral is used as the measure of approximation, after conformal mapping onto the interior of a circle; the reader can supply the details.

In Theorems 16–18 and their Corollaries we have supposed for simplicity the norm functions positive and continuous. This restriction may clearly be somewhat lightened by the methods of §5.7.

CHAPTER VI

ORTHOGONALITY AND LEAST SQUARES

§6.1. Orthogonal functions and least squares

Let C be a rectifiable Jordan arc or curve and let each of the finite or infinite set of functions $p_0(z), p_1(z), \cdots$ belong to the class L^2 on C, that is to say, let each function $p_k(z)$ be integrable (Lebesgue) together with its square on C.* The set of functions $p_k(z)$ is said to be *orthogonal* on C if we always have

$$\int_C p_k(z)\, \overline{p_n(z)}\, |\, dz\, | = 0, \qquad k \neq n,$$

and is said to be *normal* on C if we have

$$\int_C p_k(z)\, \overline{p_k(z)}\, |\, dz\, | = 1 .$$

If the functions $p_0(z),\ p_1(z),\ \cdots$ are normal and orthogonal on C, and if $f(z)$ belongs to L^2 on C, then

$$(1) \qquad f(z) \sim a_0 p_0(z) + a_1 p_1(z) + a_2 p_2(z) + \cdots, \qquad a_k = \int_C f(z)\, \overline{p_k(z)}\, |\, dz\, |,$$

is called the *formal expansion* of $f(z)$ on C in terms of the functions $p_k(z)$. Here the sign $\sim$ is used simply to denote formal correspondence. If the function $f(z)$ is equal on C to some series $\sum_{k=0}^{\infty} b_k p_k(z)$, and if it is allowable to integrate that series over C term by term after multiplication through by $\overline{p_k(z)}\, |\, dz\, |$ (it is sufficient if the given series converges uniformly on C), then the coefficients b_k are uniquely determined and are equal to the a_k in (1).

The coefficients a_k have another important property:

THEOREM 1. *If the functions $p_0(z), p_1(z), \cdots, p_n(z)$ are normal and orthogonal on C, and if $f(z)$ belongs to L^2 on C, then the linear combination of the functions $p_0(z), p_1(z), \cdots, p_n(z)$ which approximates best to $f(z)$ on C in the sense of least squares is given by*

$$(2) \qquad s_n(z) = a_0 p_0(z) + a_1 p_1(z) + \cdots + a_n p_n(z), \qquad a_k = \int_C f(z)\, \overline{p_k(z)}\, |\, dz\, |.$$

The linear combination which yields best approximation to $f(z)$ on C is the linear combination $\sum_{k=0}^{n} \lambda_k p_k(z)$ in which the λ_k are chosen so as to minimize the integral

* We frequently use the fact that the sum of two functions of class L^2 also belongs to L^2, and that the product of two functions of class L^2 is integrable in the sense of Lebesgue.

$$\text{(3)} \quad \int_C |f(z) - \sum \lambda_k p_k(z)|^2 |dz| = \int_C [f - \sum \lambda_k p_k][\bar{f} - \sum \bar{\lambda}_k \bar{p}_k] |dz|$$
$$= \int_C f\bar{f} |dz| - \sum \lambda_k \int_C \bar{f} p_k |dz| - \sum \bar{\lambda}_k \int_C f \bar{p}_k |dz| + \sum \lambda_k \bar{\lambda}_k$$
$$= \int_C f\bar{f} |dz| - \sum a_k \bar{a}_k + \sum (a_k - \lambda_k)(\bar{a}_k - \bar{\lambda}_k).$$

This last expression, considered as a function of the λ_k, is clearly a minimum when and only when we have $\lambda_k = a_k$, $k = 0, 1, 2, \cdots, n$.

COROLLARY 1. *In Theorem 1, the measure of approximation to $f(z)$ on C of the linear combination $s_n(z)$ of best approximation is the non-negative number*

$$\int_C |f(z) - s_n(z)|^2 |dz| = \int_C |f(z)|^2 |dz| - \sum_{k=0}^{n} |a_k|^2.$$

The obvious fact that this number is non-negative is known as *Bessel's inequality*; the inequality implies at once

COROLLARY 2. *If the functions $p_0(z), p_1(z), \cdots$ are normal and orthogonal on C, and if the function $f(z)$ belongs to L^2 on C, then the series*

$$\sum_{k=0}^{\infty} |a_k|^2, \qquad a_k = \int_C f(z) \overline{p_k(z)} |dz|$$

is convergent.

COROLLARY 3. *In Theorem 1, the difference $f(z) - s_n(z)$ is orthogonal to each of the functions $p_0(z), p_1(z), \cdots, p_n(z)$; this property completely characterizes the linear combination $s_n(z)$.*

Corollary 3 may be verified directly from the definition of orthogonality, or may be proved by Corollary 1.

Theorem 1 is noteworthy not merely in giving a simple formula for the linear combination of best approximation in the sense of least squares, but also in the fact that the coefficients $\lambda_k = a_k$ for best approximation by the sum $\sum_{k=0}^{n} \lambda_k p_k(z)$ do not depend on n, provided merely $n \geqq k$.

It is a well known fact that the functions $1, z, z^2, \cdots$ are mutually orthogonal on an arbitrary circle C: $|z| = R$ whose center is the origin. We have $(k \neq n)$

$$\int_C z^k \bar{z}^n |dz| = \int_C z^k \frac{R^{2n}}{z^n} \frac{R\, dz}{iz} = \frac{R^{2n+1}}{i} \int_C z^{k-n-1} dz = 0.$$

Consequently, if $f(z)$ is an arbitrary function analytic on and within C, then the polynomial

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_k = \frac{1}{2\pi R^{2k+1}} \int_C f(z)\, \bar{z}^k \,|\,dz\,| = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}}\, dz$$

is both the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least squares, and the sum of the first $n + 1$ terms of the Taylor development of $f(z)$ about the origin.

§6.2. Orthogonalization .

A finite or infinite set of functions $q_0(z)$, $q_1(z)$, $q_2(z)$, $\cdots$ of class L^2 on the rectifiable arc or curve C is said to be *linearly independent* on C provided an equation of the form

$$\int_C |\, b_0 q_0(z) + b_1 q_1(z) + \cdots + b_n q_n(z)\,|^2 \,|\,dz\,| = 0 \tag{4}$$

implies the vanishing of every b_k. It is clear that no function $q_k(z)$ of a linearly independent set can be a null function (i.e., zero almost everywhere on C). *If the $q_k(z)$ are normal and orthogonal on C, they are necessarily linearly independent on C,* for the integral in the left-hand member of (4) is precisely $|\,b_0\,|^2 + |\,b_1\,|^2 + \cdots + |\,b_n\,|^2$.

If the functions $q_0(z)$, $q_1(z)$, $\cdots$, $q_n(z)$ are of class L^2 on C, and if equation (4) is valid with the b_k not all zero, then those functions are said to be *linearly dependent* on C.

THEOREM 2. *If the finite or infinite set of functions $q_0(z)$, $q_1(z)$, $q_2(z)$, $\cdots$ of class L^2 on C is linearly independent on C, then there exists a set of functions $p_0(z)$, $p_1(z)$, $p_2(z)$, $\cdots$ which are normal and orthogonal on C; moreover $p_n(z)$ is a linear combination of the functions $q_0(z)$, $q_1(z)$, $\cdots$, $q_n(z)$, and $q_n(z)$ is a linear combination of the functions $p_0(z)$, $p_1(z)$, $\cdots$, $p_n(z)$.*

Let us define $t_0(z)$ as identically equal to $q_0(z)$, and let us set

$$p_0(z) = t_0(z)\left[\int_C |\, t_0(w)\,|^2 \,|\,dw\,|\right]^{-1/2},$$

$$t_1(z) = q_1(z) - p_0(z)\int_C q_1(w)\,\overline{p_0(w)}\,|\,dw\,|,$$

$$p_1(z) = t_1(z)\left[\int_C |\, t_1(w)\,|^2 \,|\,dw\,|\right]^{-1/2},$$

$$t_2(z) = q_2(z) - p_0(z)\int_C q_2(w)\,\overline{p_0(w)}\,|\,dw\,| - p_1(z)\int_C q_2(w)\,\overline{p_1(w)}\,|\,dw\,|,$$

$$p_2(z) = t_2(z)\left[\int_C |\, t_2(w)\,|^2 \,|\,dw\,|\right]^{-1/2},$$

. . . . ,

$$t_n(z) = q_n(z) - p_0(z) \int_C q_n(w)\,\overline{p_0(w)}\,|\,dw\,| - p_1(z) \int_C q_n(w)\,\overline{p_1(w)}\,|\,dw\,|$$

$$- \cdots - p_{n-1}(z) \int_C q_n(w)\,\overline{p_{n-1}(w)}\,|\,dw\,|,$$

$$p_n(z) = t_n(z) \left[\int_C |\, t_n(w)\,|^2\,|\,dw\,| \right]^{-1/2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

In these defining equations, we have defined $p_0(z)$ as a constant multiple of $q_0(z)$ which is normal on C. The function $t_1(z)$ is equal to $q_1(z)$ diminished by the formal expansion of $q_1(z)$ in terms of $p_0(z)$, hence (Corollary 3) $t_1(z)$ is orthogonal to $p_0(z)$. The function $p_1(z)$ is a constant multiple of $t_1(z)$ which is normal on C, hence $p_1(z)$ is also orthogonal to $p_0(z)$. Similarly, the function $t_n(z)$ is equal to $q_n(z)$ diminished by the formal expansion of $q_n(z)$ in terms of $p_0(z)$, $p_1(z)$, $\cdots$, $p_{n-1}(z)$; hence (Corollary 3) $t_n(z)$ is orthogonal to $p_0(z)$, $p_1(z)$, $\cdots$, $p_{n-1}(z)$, and therefore $p_n(z)$ (a constant multiple of $t_n(z)$ which is normal on C) is also orthogonal to these functions. Thus each function $p_n(z)$ is orthogonal to the functions $p_0(z)$, $p_1(z)$, $\cdots$, $p_{n-1}(z)$, and hence the set $p_n(z)$ is normal and orthogonal. The function $p_n(z)$ is obviously a linear combination of the functions $p_0(z)$, $p_1(z)$, $\cdots$, $p_{n-1}(z)$, $q_n(z)$, hence (since $p_0(z)$ is a multiple of $q_0(z)$) a linear combination of the functions $q_0(z)$, $q_1(z)$, $\cdots$, $q_n(z)$; the function $q_n(z)$ is obviously a linear combination of the functions $p_0(z)$, $p_1(z)$, $\cdots$, $p_n(z)$.

In order to justify the procedure just used, it remains merely to show that no function $t_n(z)$ is a null function. The function $t_0(z)$ is identical with the given function $q_0(z)$, hence not a null function. If n is the smallest subscript for which $t_n(z)$ is a null function, it follows from the definition of $t_n(z)$ that the functions $p_0(z)$, $p_1(z)$, $\cdots$, $p_{n-1}(z)$, $q_n(z)$ are linearly dependent. In the corresponding relation of form (4), the coefficient of $q_n(z)$ is different from zero; hence the functions $q_0(z), q_1(z), \cdots, q_n(z)$ are also linearly dependent, contrary to hypothesis.

The actual formulas for the $p_n(z)$ directly in terms of the $q_k(z)$ are not difficult to derive, and are well known; see for instance Kowalewski [1909].

Since every linear combination of the functions $q_0(z)$, $q_1(z)$, $\cdots$, $q_n(z)$ is a linear combination of the functions $p_0(z)$, $p_1(z)$, $\cdots$, $p_n(z)$, and conversely, it follows that the linear combination of the former set of functions of best approximation to an arbitrary function $f(z)$ of L^2 on C in the sense of least squares is identical with the linear combination of the latter set of functions of best approximation to $f(z)$ on C in the sense of least squares, for which a convenient formula is given in Theorem 1. In order to study approximation by the functions $q_k(z)$ it is sufficient to study approximation by the functions $p_k(z)$. Herein lies the chief importance of orthogonalization.

A simple interpretation can be given for the orthogonalization process. The function $t_n(z)$ is (Theorem 1) the function of the form

$$q_n(z) + c_0 p_0(z) + c_1 p_1(z) + \cdots + c_{n-1} p_{n-1}(z)$$

whose norm (integral over C of the square of the modulus) is least, and hence also the function of the form

$$q_n(z) + c_0' q_0(z) + c_1' q_1(z) + \cdots + c_{n-1}' q_{n-1}(z)$$

whose norm is least. Let us show that the function $p_n(z)$ is the normalized function (i.e., of norm unity) of the form

$$d_n q_n(z) + d_0 p_0(z) + d_1 p_1(z) + \cdots + d_{n-1} p_{n-1}(z)$$

or

$$d_n q_n(z) + d_0' q_0(z) + d_1' q_1(z) + \cdots + d_{n-1}' q_{n-1}(z)$$

such that d_n is real and maximum. Denote by d_n^0 the coefficient of $q_n(z)$ in $p_n(z)$. Any function of the prescribed form which is a multiple of $p_n(z)$ whose coefficient of $q_n(z)$ is greater than d_n^0 has a norm greater than unity, and the norm of any other function of the prescribed form whose coefficient of $q_n(z)$ is greater than d_n^0 is still greater.

Suppose a given function $q_n(z)$ is *linearly dependent* on C on the functions $q_0(z), q_1(z), \cdots, q_{n-1}(z)$, all of class L^2, in the sense that numbers b_k exist such that (4) is valid, with b_n different from zero. Then any measure of approximation of the form

$$\int_C | f(z) - a_0 q_0(z) - a_1 q_1(z) - \cdots - a_n q_n(z) |^2 \, | dz |$$

can also be written in the form

$$\int_C \Big| f(z) - a_0 q_0(z) - a_1 q_1(z) - \cdots - a_{n-1} q_{n-1}(z) + \frac{a_n}{b_n} [b_0 q_0(z) + b_1 q_1(z) + \cdots + b_{n-1} q_{n-1}(z)] \Big|^2 \, | dz | ,$$

since the functions $q_n(z)$ and

$$-\frac{1}{b_n} [b_0 q_0(z) + b_1 q_1(z) + \cdots + b_{n-1} q_{n-1}(z)]$$

are equal almost everywhere on C. Hence approximation on C to an arbitrary function $f(z)$ of class L^2 by a linear combination of the functions $q_0(z), q_1(z), \cdots, q_n(z)$ is equivalent to suitably chosen approximation to $f(z)$ on C by a linear combination of the functions $q_0(z), q_1(z), \cdots, q_{n-1}(z)$, in the sense that the measure of approximation in the latter case is the same as the measure of approximation in the former case. In this sense we are justified henceforth in suppressing any function $q_n(z)$ linearly dependent on the preceding functions $q_k(z)$; that is to say, we are justified in assuming the functions of the given set $q_k(z)$ linearly independent, and the given set $q_k(z)$ can be orthogonalized.

§6.3. Riesz-Fischer theory

Corollary 2 to Theorem 1 immediately suggests the question of the existence of a function $f(z)$ when the coefficients a_k are given arbitrarily. We shall prove the Riesz-Fischer theorem:

THEOREM 3. *If the functions $p_0(z)$, $p_1(z)$, $p_2(z)$, $\cdots$ are normal and orthogonal on C, and if the numbers a_0, a_1, a_2, $\cdots$ are given such that $\sum | a_k |^2$ converges, then there exists a function $f(z)$ of class L^2 on C such that we have*

$$(5) \qquad a_k = \int_C f(z) \overline{p_k(z)} \, | dz | , \qquad k = 0, 1, 2, \cdots .$$

One such function $f(z)$ is the limit in the mean on C of the series

$$(6) \qquad \sum_{k=0}^{\infty} a_k p_k(z) .$$

A natural method of proving the existence of such a function $f(z)$ is to study the series (6). This series need not be convergent, but is nevertheless *convergent in the mean* on C. A series $s_0(z) + [s_1(z) - s_0(z)] + [s_2(z) - s_1(z)] + \cdots$ or the corresponding sequence $s_k(z)$ is said to be *convergent in the mean* on C provided we have

$$(7) \qquad \lim_{m,\, n \to \infty} \int_C | s_m(z) - s_n(z) |^2 \, | dz | = 0 .$$

Such a series or sequence is said *to converge in the mean on C to the limit function* $f(z)$ provided we have

$$(8) \qquad \lim_{n \to \infty} \int_C | f(z) - s_n(z) |^2 \, | dz | = 0 .$$

It is by no means obvious that condition (7) implies the existence of a function $f(z)$ of class L^2 such that (8) is satisfied; the proposition is true, however:

LEMMA. *If the sequence of functions $s_k(z)$ each of class L^2 on C converges in the mean on C, then there exists a function $f(z)$ of class L^2 to which the sequence $s_k(z)$ converges in the mean on C.*

This lemma is due to Weyl, but the present formulation of the proof is essentially due to von Neumann.

Let an arbitrary positive ϵ be given. Choose $N = N(\epsilon)$ so that

$$(9) \qquad \int_C | s_m(z) - s_n(z) |^2 \, | dz | < \epsilon, \qquad \text{whenever } m > N \,, n > N .$$

Choose $N_p > N(1/8^p)$, $N_{p+1} > N_p$. It then follows from (9) that the set of points on which $| s_{N_{p+1}}(z) - s_{N_p}(z) | \geqq 1/2^p$ is of measure not greater than $1/2^p$. Hence *all* the inequalities

$$| s_{N_{p+1}}(z) - s_{N_p}(z) | < 1/2^p, \qquad | s_{N_{p+2}}(z) - s_{N_{p+1}}(z) | < 1/2^{p+1}, \ldots$$

are valid on a set E_p such that $\text{meas}(C-E_p) \leqq \sum_{\alpha=p}^{\alpha=\infty} 1/2^\alpha = 1/2^{p-1}$. The set E_p is a subset of E_{p+1} for every p. Then the sequence $s_{N_n}(z)$ converges uniformly on every E_p, for the condition $m > n \geqq p$ implies on E_p

$$| s_{N_m}(z) - s_{N_n}(z) | \leqq \sum_{\alpha=n}^{m-1} | s_{N_{\alpha+1}}(z) - s_{N_\alpha}(z) | \leqq \sum_{\alpha=n}^{m-1} 1/2^\alpha < 1/2^{n-1},$$

which approaches zero with $1/n$. The sequence $s_{N_n}(z)$ therefore converges at every point of the set $E_0 = E_1 + E_2 + E_3 + \cdots$, and we have $\text{meas}(C-E_0) = 0$.

We introduce the allowable definition

$$f(z) = \lim_{n\to\infty} s_{N_n}(z), \qquad z \text{ on } E_0,$$

$$f(z) = 0, \qquad z \text{ on } C - E_0;$$

we shall prove that this function satisfies the conditions of the Lemma. We clearly have for $m > N(\epsilon)$, $N_p > N(\epsilon)$,

$$\int_{E_k} | s_m(z) - s_{N_p}(z) |^2 \, | dz | \leqq \int_C | s_m(z) - s_{N_p}(z) |^2 \, | dz | < \epsilon.$$

The sequence $s_{N_p}(z)$ converges uniformly to $f(z)$ on E_k, whence we conclude by the Schwarz inequality [§5.7, inequality (30)] or the triangle inequality (§11.4)

$$\int_{E_k} | s_m(z) - f(z) |^2 \, | dz | \leqq \epsilon, \qquad m > N(\epsilon),$$

and this inequality holds for every k. The set E_k is monotonically non-decreasing as k increases, so we may allow k to become infinite. There follows the inequality

$$\int_C | s_m(z) - f(z) |^2 \, | dz | \leqq \epsilon \tag{10}$$

provided merely $m > N(\epsilon)$; inequality (10) is equivalent to (8), and the existence (just established implicitly) of the integral in (10) implies that $f(z)$ is of class L^2 on C; the Lemma is established.

COROLLARY. *The function $f(z)$ of the Lemma is essentially unique on C.*

If another such function exists, say $f_1(z)$, we have for arbitrary positive ϵ and suitably chosen n

$$\int_C | s_n(z) - f(z) |^2 \, | dz | < \epsilon, \qquad \int_C | s_n(z) - f_1(z) |^2 \, | dz | < \epsilon.$$

By §5.2, inequality (10) we have

$$\int_C | f(z) - f_1(z) |^2 \, | dz | < 4\epsilon,$$

so $f(z)$ and $f_1(z)$ can differ on C at most on a set of measure zero, which is to say that $f(z)$ and $f_1(z)$ are essentially identical on C.

We return now to the proof of Theorem 3. The convergence of the given series $\sum |a_k|^2$ implies the condition $(m > n)$

$$\lim_{m,n\to\infty} [|a_{n+1}|^2 + |a_{n+2}|^2 + \cdots + |a_m|^2] = 0,$$

which is precisely the condition (7) for the convergence in the mean on C of the series (6). It remains to show that the function $f(z)$, whose existence is now asserted by the Lemma and which belongs to L^2, satisfies equations (5). By the Schwarz inequality [§5.7 inequality (30), $\alpha = 1/2$] we can write

$$(11)\quad \left| \int_C \overline{p_k(z)}\, [f(z) - s_n(z)]\, |dz| \right|^2 \leqq \int_C |f(z) - s_n(z)|^2\, |dz| \cdot \int_C |p_k(z)|^2\, |dz|.$$

The second factor on the right is unity, and the first factor on the right approaches zero with $1/n$. The expression whose modulus is squared in the left-hand member has the value (for $n \geqq k$)

$$\int_C f(z)\, \overline{p_k(z)}\, |dz| - a_k,$$

which by (11) approaches zero with $1/n$ and therefore has the constant value zero. That is to say, equation (5) is established, and hence also Theorem 3.

It is clear that in (11) we may replace $\overline{p_k(z)}$ by any function of class L^2, with a corresponding conclusion:

COROLLARY 1. *If $\phi(z)$ is of class L^2 on C, then we have*

$$\lim_{n\to\infty} \int_C \phi(z)\, s_n(z)\, |dz| = \int_C \phi(z)\, f(z)\, |dz|.$$

Indeed, this equation is true whenever the sequence $s_n(z)$ of class L^2 on C converges to $f(z)$ in the mean on C.

In a similar way we may write

$$\left| \int_C \overline{s_n(z)}\, [f(z) - s_n(z)]\, |dz| \right|^2 \leqq \int_C |f(z) - s_n(z)|^2\, |dz| \cdot \int_C |s_n(z)|^2\, |dz|.$$

The second factor on the right approaches $\sum_{k=0}^{\infty} |a_k|^2$, so the left-hand member approaches zero. But the left-hand member approaches (by Corollary 1) the square of the absolute value of

$$\int_C |f(z)|^2\, |dz| - \sum_{k=0}^{\infty} |a_k|^2,$$

so we have established

COROLLARY 2. *For the function $f(z)$ of Theorem 3 exhibited by the Lemma we have*

$$\int_C | f(z) |^2 \, | dz | = \sum_{k=0}^{\infty} | a_k |^2.$$

Corollary 2 is a special case of a more general result that can be proved in a similar way: whenever the sequence $s_n(z)$ of class L^2 on C converges to $f(z)$ in the mean on C, then we have

$$\lim_{n\to\infty} \int_C | s_n(z) |^2 \, | dz | = \int_C | f(z) |^2 \, | dz | .$$

It is naturally possible for the function $f(z)$ of Theorem 3 not to be unique For instance, if the function $P(z)$ of class L^2 is orthogonal to all the $p_k(z)$ on C, then by Corollary 1 the function $f(z) + \lambda P(z)$, where λ is constant, also satisfies the requirements of Theorem 3. The particular function exhibited in the proof of Theorem 3 has, however, a characteristic property.

COROLLARY 3. *If $f(z)$ is the limit in the mean on C of series* (6), *under the hypothesis of Theorem* 3, *and if $F(z)$ is any function of class L^2 differing from $f(z)$ on C on a set of positive measure and such that*

$$\int_C F(z) \overline{p_k(z)} \, | dz | = a_k, \qquad k = 0, 1, 2, \cdots, \tag{12}$$

then we have

$$\int_C | f(z) |^2 \, | dz | < \int_C | F(z) |^2 \, | dz | .$$

Otherwise expressed, of all functions $F(z)$ of class L^2 such that (12) is valid, the function $f(z)$ is the essentially unique one of minimum *norm* on C

By (5) and (12) we have

$$\int_C [F(z) - f(z)] \overline{p_k(z)} \, | dz | = 0 , \qquad \int_C [F(z) - f(z)] \overline{s_n(z)} \, | dz | = 0 ,$$

whence by Corollary 1,

$$\int_C [F(z) - f(z)] \overline{f(z)} \, | dz | = 0 , \qquad \int_C [\overline{F(z)} - \overline{f(z)}] f(z) \, | dz | = 0 .$$

Since $F(z)$ differs from $f(z)$ on C on a set of positive measure, we may write

$$0 < \int_C [F(z) - f(z)][\overline{F(z)} - \overline{f(z)}] \, | dz | = \int_C | F(z) |^2 \, | dz | - \int_C | f(z) |^2 \, | dz | ,$$

as we were to prove.

We can state immediately from the inequalities just used

COROLLARY 4. *Under the conditions of Corollary* 3 *we have*

$$\int_C |F(z) - f(z)|^2 \, |dz| = \int_C |F(z)|^2 \, |dz| - \int_C |f(z)|^2 \, |dz|.$$

In particular if $F_n(z)$ is a function which satisfies the hypothesis made on $F(z)$, and if

$$\lim_{n\to\infty} \int_C |F_n(z)|^2 \, |dz| = \int_C |f(z)|^2 \, |dz|,$$

then the sequence $F_n(z)$ converges to $f(z)$ in the mean on C.

A consequence of Corollaries 2 and 3 is

COROLLARY 5. *If $F(z)$ is of class L^2 on C, if equations* (12) *are valid, and if we have*

$$\int_C |F(z)|^2 \, |dz| = \sum_{k=0}^{\infty} |a_k|^2,$$

then $F(z)$ is equal almost everywhere on C to the function $f(z)$ exhibited as the limit in the mean of series (6); *series* (6) *converges in the mean to $F(z)$ on C.*

§6.4. Closure

THEOREM 4. *Let the functions $p_k(z)$ be normal and orthogonal on C, and let $F(z)$ be of class L^2 on C. A necessary and sufficient condition that we have*

$$\int_C |F(z)|^2 \, |dz| = \sum_{k=0}^{\infty} |a_k|^2, \qquad a_k = \int_C F(z) \, \overline{p_k(z)} \, |dz|, \tag{13}$$

is the following: Given an arbitrary positive ϵ, there exist numbers $n, b_0, b_1, \cdots, b_n$ such that we have

$$\int_C |F(z) - b_0p_0(z) - b_1p_1(z) - \cdots - b_np_n(z)|^2 \, |dz| < \epsilon.$$

If this condition is satisfied, we have a fortiori (by Theorem 1)

$$\int_C |F(z) - a_0p_0(z) - a_1p_1(z) - \cdots - a_np_n(z)|^2 \, |dz| < \epsilon,$$

$$\int_C |F(z)|^2 \, |dz| - \sum_{k=0}^{n} |a_k|^2 < \epsilon.$$

This summation increases or remains unchanged when n increases, and the entire left-hand member is never negative (Theorem 1, Corollary 1). Equation (13)

follows immediately. We have proved incidentally (by Theorem 3, Corollary 5) the

COROLLARY. *If the condition of Theorem 4 is satisfied, then $F(z)$ is the limit in the mean of the series $\sum_{k=0}^{\infty} a_k p_k(z)$.*

Conversely, if (13) is satisfied and if ϵ is given we have for suitably chosen n

$$\epsilon > \int_C |F(z)|^2 |dz| - \sum_{k=0}^{n} |a_k|^2$$

$$= \int_C |F(z) - a_0 p_0(z) - a_1 p_1(z) - \cdots - a_n p_n(z)|^2 |dz|,$$

and the proof is complete.

Equation (13) is called *Parseval's equation*, or the *equation of closure.*

Another form of Parseval's equation is sometimes useful: *If the function $F(z)$ satisfies the hypothesis of Theorem 4 and if $G(z)$ belongs to L^2, then we have*

$$\int_C F(z)\,\overline{G(z)}\,|dz| = \sum_{k=0}^{\infty} a_k \bar{c}_k, \qquad c_k = \int_C G(z)\,\overline{p_k(z)}\,|dz|.$$

The proof is immediate, for $F(z)$ is the limit in the mean of its formal expansion:

$$\int_C F(z)\,\overline{G(z)}\,|dz| = \sum_{k=0}^{\infty} \int_C a_k p_k(z)\,\overline{G(z)}\,|dz| = \sum_{k=0}^{\infty} a_k \bar{c}_k;$$

we apply Theorem 3, Corollary 1.

Let $\mathfrak{H}$ be a class of functions on C which belong to L^2 and perhaps satisfy other conditions. The class $\mathfrak{H}$ is said to be *linear* if $\lambda_1\phi_1(z) + \lambda_2\phi_2(z)$ belongs to $\mathfrak{H}$ whenever $\phi_1(z)$ and $\phi_2(z)$ belong to $\mathfrak{H}$, where λ_1 and λ_2 are arbitrary constants. The class $\mathfrak{H}$ is said to be *closed* if the limit in the mean on C of every sequence of functions of $\mathfrak{H}$ convergent in the mean is also a function of $\mathfrak{H}$. The set of functions $p_0(z)$, $p_1(z)$, $\cdots$ is said to be *closed* or *complete* on C with respect to the functions of class $\mathfrak{H}$ (linear and closed) provided the functions $p_k(z)$ belong to $\mathfrak{H}$ and provided no function of $\mathfrak{H}$ not a null function is orthogonal to all the $p_k(z)$.

THEOREM 5. *If the normal and orthogonal set $p_k(z)$ is closed on C with respect to all functions of class $\mathfrak{H}$ (a linear closed subset of L^2), then for any function $F(z)$ of class $\mathfrak{H}$ equation* (13) *is valid.*

Conversely, if equation (13) *holds for every function $F(z)$ of class $\mathfrak{H}$ (a subset of class L^2 but not necessarily linear or closed), then there exists no function of class $\mathfrak{H}$ not a null function orthogonal to all the $p_k(z)$. In particular if the functions $p_k(z)$ belong to $\mathfrak{H}$, and if $\mathfrak{H}$ is linear and closed, then the set $\{p_k(z)\}$ is closed on C with respect to the functions of class $\mathfrak{H}$.*

Let the set $\{p_k(z)\}$ be closed on C with respect to class $\mathfrak{H}$, and let $F(z)$ be an arbitrary function of $\mathfrak{H}$. Let $f(z)$ denote the function (necessarily of class $\mathfrak{H}$) which is the limit in the mean on C of the series

$$\sum_{k=0}^{\infty} a_k p_k(z), \qquad a_k = \int_C F(z)\, \overline{p_k(z)}\, |\, dz\, |.$$

Then the function $F(z) - f(z)$ belongs to $\mathfrak{H}$, is orthogonal to all the $p_k(z)$, hence is a null function. Equation (13) is a consequence of Corollary 2 to Theorem 3.

Conversely, let us suppose (13) to hold for every $F(z)$ belonging to $\mathfrak{H}$. If a function $F(z)$ of $\mathfrak{H}$ is orthogonal to all the $p_k(z)$ on C, then all the coefficients a_k in (13) vanish, so we have

$$\int_C |\, F(z)\, |^2\, |\, dz\, | = 0,$$

and $F(z)$ is a null function as we were to prove.

Corollary 1. *Let the set $p_k(z)$ normal and orthogonal on C belong to some linear and closed class $\mathfrak{H}$. A necessary and sufficient condition that the formal expansion in the $p_k(z)$ of every function $F(z)$ of $\mathfrak{H}$ converge in the mean to $F(z)$ on C is that the set $p_k(z)$ be closed on C with respect to the functions of class $\mathfrak{H}$.*

The condition is necessary, for if the formal expansion of $F(z)$ of $\mathfrak{H}$ converges in the mean on C to $F(z)$, the equation of closure follows from Corollary 2 to Theorem 3 and the closure of the set $p_k(z)$ with respect to $\mathfrak{H}$ follows from the second part of Theorem 5.

The condition is sufficient, for otherwise the formal expansion of some $F(z)$ of $\mathfrak{H}$ converges in the mean on C to some function $f(z)$ of $\mathfrak{H}$ which is not equal almost everywhere to $F(z)$. The difference $F(z) - f(z)$ belongs to $\mathfrak{H}$ and is not a null function; its formal expansion in the $p_k(z)$ vanishes identically, so (13) is not satisfied, in contradiction to the first part of Theorem 5.

Corollary 2. *Let the set $p_k(z)$ be normal and orthogonal on C, and let $\mathfrak{H}$ be the set of all functions $f(z)$ on C each of which is the limit in the mean of a series $\sum_{k=0}^{\infty} a_k p_k(z)$, with $\sum_{k=0}^{\infty} |\, a_k\, |^2$ convergent. That is to say, $\mathfrak{H}$ is the set of all functions $f(z)$ on C for which we have*

$$\int_C |\, f(z)\, |^2\, |\, dz\, | = \sum_{k=0}^{\infty} |\, a_k\, |^2, \qquad a_k = \int_C f(z)\, \overline{p_k(z)}\, |\, dz\, |.$$

Then the set $\mathfrak{H}$ is linear and closed. The set $p_k(z)$ is closed on C with respect to functions of class $\mathfrak{H}$.

The identity of these two conditions on $f(z)$ follows from Theorem 3, Corollaries 2 and 5. Moreover, the first of these conditions is equivalent to the con-

dition that $f(z)$ should be the limit in the mean on C of its own formal deve ment in terms of the set $p_k(z)$.

The class $\mathfrak{H}$ is called the *closed extension* on C of the set $p_k(z)$; this concept appears frequently in the sequel. We proceed to the proof of the theorem.

The linearity of the set $\mathfrak{H}$ is obvious [§5.2, inequality (10)], and the closure is not difficult to establish. Let the set of functions $f_1(z), f_2(z), \cdots$ of $\mathfrak{H}$ converge in the mean on C to the function $f(z)$; it remains to show that $f(z)$ belongs to $\mathfrak{H}$. Let $\epsilon > 0$ be given. There exists $f_m(z)$ such that

$$\int_C | f(z) - f_m(z) |^2 \, | dz | < \epsilon/4 .$$

By Theorem 4 there exist numbers $n, b_0, b_1, \cdots, b_n$ such that

$$\int_C | f_m(z) - b_0p_0(z) - b_1p_1(z) - \cdots - b_np_n(z) |^2 \, | dz | < \epsilon/4 .$$

Consequently we have by §5.2, inequality (10)

$$\int_C | f(z) - b_0p_0(z) - b_1p_1(z) - \cdots - b_np_n(z) |^2 \, | dz | < \epsilon ,$$

and it follows from Theorem 4 that $f(z)$ belongs to $\mathfrak{H}$.

The closure on C of the set $p_k(z)$ with respect to functions of class $\mathfrak{H}$ is a consequence of Corollary 1.

We state for reference a result which is essentially contained in the proof of Corollary 2.

Corollary 3. *Let each of the two sets $p_k(z)$ and $r_k(z)$ be normal and orthogonal on C. If each function $r_n(z)$ belongs to the closed extension of the $p_k(z)$, and if each function $p_n(z)$ belongs to the closed extension of the $r_k(z)$, then the two sets $p_k(z)$ and $r_k(z)$ have the same closed extension.*

We have already proved that corresponding to any normal orthogonal set $p_k(z)$ there exists a linear closed set $\mathfrak{H}$, the closed extension of the set $p_k(z)$. As a consequence of the discussion of §6.2, we may start with any set $q_0(z)$, $q_1(z)$, $q_2(z)$, $\cdots$ of functions of class L^2 on C, whether orthogonal or not. Let us prove

Corollary 4. *The set $\mathfrak{H}$ of functions $f(z)$ each of which is the limit in the mean on C of a sequence of linear combinations of the functions $q_k(z)$ is linear and closed.*

Indeed, the set $\mathfrak{H}$ is identical with the set $\mathfrak{H}'$ of functions $f(z)$ each of which is the limit in the mean of a sequence of linear combinations of functions $p_k(z)$, where the functions $p_k(z)$ are found (§6.2) by orthogonalization and normalization of the set $q_k(z)$. Any function $q_n(z)$ which is linearly dependent on the functions $q_0(z)$, $q_1(z)$, $\cdots$, $q_{n-1}(z)$ may obviously be omitted from the original enumeration of the $q_k(z)$, or may be omitted in the process of orthogonalization;

ch are limits in the mean is independent of such omission;
the end of §6.2. The closure of the set $\mathfrak{H}'$ and hence of
Corollary 2. The set $\mathfrak{H}$ may be called the *closed extension* also
$_k(z)$.

By combining Corollaries 3 and 4 we have now

COROLLARY 5. *Let $\mathfrak{H}$ and $\mathfrak{H}'$ be the closed extensions on C of the respective sets of functions $q_k(z)$ and $q'_k(z)$ of class L^2. If each $q_k(z)$ belongs to $\mathfrak{H}'$, then $\mathfrak{H}$ is contained in $\mathfrak{H}'$; if each $q_k(z)$ belongs to $\mathfrak{H}'$ and each $q'_k(z)$ belongs to $\mathfrak{H}$, then $\mathfrak{H}$ and $\mathfrak{H}'$ are identical.*

We prove a further result concerning classes $\mathfrak{H}$ essentially a generalization of Theorem 1:

THEOREM 6. *If $F(z)$ is of class L^2 on C, if $f(z)$ is the limit in the mean of the formal expansion of $F(z)$ in terms of the normal and orthogonal functions $p_k(z)$, and if $\mathfrak{H}$ denotes the closed extension on C of the set $p_k(z)$, then $f(z)$ is the essentially unique function of $\mathfrak{H}$ such that*

$$\int_C |F(z) - f(z)|^2 \, |dz|$$

is least. The function $f(z)$ (of class $\mathfrak{H}$) is completely characterized by the fact that $F(z) - f(z)$ is orthogonal on C to all the functions $p_k(z)$, hence orthogonal on C to all the functions of class $\mathfrak{H}$.

The proof is similar to that of Theorem 3, Corollary 3. We show that if $g(z)$ is a function of $\mathfrak{H}$ different from $f(z)$ on a set of positive measure, then we have

$$\int_C |F(z) - g(z)|^2 \, |dz| > \int_C |F(z) - f(z)|^2 \, |dz|.$$

The function $F(z) - f(z)$ is orthogonal on C to every $p_k(z)$, so we have (Corollary 2 to Theorem 5 and Corollary 1 to Theorem 3)

$$\int_C [F(z) - f(z)]\, \overline{g(z)} \, |dz| = 0, \qquad \int_C [\overline{F(z)} - \overline{f(z)}]\, g(z) \, |dz| = 0,$$

$$\int_C [F(z) - f(z)]\, \overline{f(z)} \, |dz| = 0, \qquad \int_C [\overline{F(z)} - \overline{f(z)}]\, f(z) \, |dz| = 0.$$

The obvious inequality

$$0 < \int_C |f(z) - g(z)|^2 \, |dz| = \int_C |[F(z) - f(z)] - [F(z) - g(z)]|^2 \, |dz|$$

now reduces to

$$0 < \int_C |F(z) - g(z)|^2 \, |dz| - \int_C |F(z) - f(z)|^2 \, |dz|,$$

as we were to prove.

COROLLARY 1. *Under the hypothesis of Theorem 6 and if $g(z)$ is an arbitrary function of $\mathfrak{H}$, we have*

$$\int_C |f(z) - g(z)|^2 |dz| = \int_C |F(z) - g(z)|^2 |dz| - \int_C |F(z) - f(z)|^2 |dz|.$$

In particular, if $g_n(z)$ is a function of $\mathfrak{H}$ such that

$$\lim_{n\to\infty} \int_C |F(z) - g_n(z)|^2 |dz| = \int_C |F(z) - f(z)|^2 |dz|,$$

then $g_n(z)$ converges to $f(z)$ in the mean on C.

COROLLARY 2. *Let $F(z)$ be of class L^2 on C, let $\mathfrak{H}$ be the closed extension on C of the normal orthogonal set $p_k(z)$, and let the numbers $a_{n_1}, a_{n_2}, \cdots$ be given so that $\sum_{k=1}^{\infty} |a_{n_k}|^2$ is convergent, $n_1 < n_2 < \cdots$. Denote by $\phi(z)$ the function which is the limit in the mean on C of the series $\sum_{k=1}^{\infty} a_{n_k} p_{n_k}(z)$. Then the essentially unique function $f(z)$ of class $\mathfrak{H}$ such that $\int_C f(z)\,\overline{p_{n_k}(z)}\,|dz| = a_{n_k}$ and such that*

$$\int_C |F(z) - f(z)|^2 |dz|$$

is least is $\phi(z)$ plus the limit in the mean on C of the formal development of $F(z) - \phi(z)$ [or of $F(z)$ itself] in terms of the set $p_0(z), p_1(z), \cdots, p_{n_1-1}(z), p_{n_1+1}(z), \cdots, p_{n_2-1}(z), p_{n_2+1}(z), \cdots$.

Denote by $\mathfrak{H}'$ the closed extension of this latter set of normal orthogonal functions. Any function of class $\mathfrak{H}$ whose coefficients with respect to the $p_{n_k}(z)$ are the a_{n_k} is equal to $\phi(z)$ plus some function of class $\mathfrak{H}'$; the given problem is equivalent to that of minimizing

$$\int_C |[F(z) - \phi(z)] - [f(z) - \phi(z)]|^2 |dz|$$

where $f(z) - \phi(z)$ is an arbitrary function of class $\mathfrak{H}'$. The solution of this problem is as indicated in Theorem 6

The minimizing function $f(z)$ is therefore the limit in the mean of the series $\sum_{n=0}^{\infty} c_n p_n(z)$, where we set

$$c_n = a_n \text{ if } n = n_k, \qquad c_n = \int_C F(z)\,\overline{p_n(z)}\,dz \text{ if } n \neq n_k.$$

Corollary 2 contains essentially both Theorem 6 and Corollary 3 to Theorem 3.

§6.5. Polynomial approximation to analytic functions

Let C be a rectifiable Jordan arc or curve and let $n(z)$ (not a null function) be real, non-negative, and uniformly limited on C; more generally, it is sufficient for our purposes if the symbols about to be considered have a meaning. The functions

$p_k(z)$ of class L^2 on C are said to be *orthogonal on C with respect to the weight or norm function $n(z)$* if we have

$$\int_C n(z)\, p_k(z)\, \overline{p_n(z)}\, |\, dz\, | = 0, \qquad k \neq n,$$

and are said to be *normal on C with respect to the weight function $n(z)$ if we have*

$$\int_C n(z)\, |\, p_k(z)\, |^2\, |\, dz\, | = 1.$$

The concept of orthogonality with respect to the weight function $n(z)$ has the same relation to approximation in the sense of least squares with respect to the weight function $n(z)$ as has orthogonality in the usual sense ($n(z) \equiv 1$) to approximation in the sense of least squares in the usual sense ($n(z) \equiv 1$). A set of given functions of class L^2 and linearly independent with respect to $n(z)$ can be orthogonalized and normalized on C with respect to $n(z)$. All the discussion of §§6.1–6.4 can be generalized to apply to the present situation. Indeed, the present results can be obtained from the preceding results: the functions $\{p_k(z)\}$ are orthogonal on C with respect to $n(z)$ if the functions $\{[n(z)]^{1/2}p_k(z)\}$ are orthogonal on C in the usual sense; the given functions $\{q_k(z)\}$ can be orthogonalized on C with respect to $n(z)$ by orthogonalizing the functions

$$\{[n(z)]^{1/2}q_k(z)\};$$

least-square approximation to $f(z)$ on C by the functions $q_n(z)$ with respect to the norm function $n(z)$ is equivalent to least-square approximation to the function $[n(z)]^{1/2}f(z)$ on C by the functions $[n(z)]^{1/2}q_n(z)$ in the usual sense:

$$\int_C n(z)\, |\, f(z) - b_0q_0(z) - b_1q_1(z) - \cdots - b_nq_n(z)\, |^2\, |\, dz\, |$$
$$= \int_C |\, [n(z)]^{1/2}f(z) - b_0[n(z)]^{1/2}q_0(z) - b_1[n(z)]^{1/2}q_1(z) - \cdots$$
$$- b_n[n(z)]^{1/2}q_n(z)\, |^2\, |\, dz\, |.$$

The new analogue of (5) is

$$a_k = \int_C n(z)\, f(z)\, \overline{p_k(z)}\, |\, dz\, |, \tag{14}$$

and the new analogue of (13) is (in the present notation, including (14))

$$\int_C n(z)\, |\, f(z)\, |^2\, |\, dz\, | = \sum_{k=0}^{\infty} |\, a_k\, |^2; \tag{15}$$

we leave to the reader the verification of all these facts.

The functions $1, z, z^2, \cdots$ are clearly linearly independent on an arbitrary rectifiable Jordan arc or curve C, or on γ: $|\, w\, | = 1$ after conformal mapping as

in §5.4, even with a positive continuous norm function. For instance, the relation

$$\int_C n(z) \mid b_0 + b_1 z + \cdots + b_m z^m \mid^2 \mid dz \mid = 0$$

implies

$$b_0 + b_1 z + \cdots + b_m z^m \equiv 0, \qquad z \text{ on } C,$$

and hence implies the vanishing of all the coefficients b_j; in the situation corresponding to §5.4 the same conclusion is an immediate consequence of the Lemma of §5.4. Then the functions $1, z, z^2, \cdots$ can be orthogonalized and normalized on C or on γ. The normal orthogonal functions $p_k(z)$ thus obtained are polynomials in z of respective degrees $k = 0, 1, 2, \cdots$.

The general study of polynomials orthogonal on a curve C was first undertaken by Szegö, in a brilliant paper [1921]. Szegö studies asymptotic properties of the polynomials and coefficients, convergence and divergence of the expansion of analytic functions, roots of the orthogonal polynomials, etc.; Szego proves in particular the major part of §5.2, Theorem 3 for the case $n(z) \equiv 1$, $p = 2$, C an analytic Jordan curve, and also studies [1921b] in detail the case $p = 2$, $n(z)$ arbitrary, C the unit circle or a finite line segment.

We have thus far (§§6.1–6.4) considered orthogonality with respect to a norm function (which may or may not be unity) as measured by a *line* integral. The entire theory discussed is applicable with only formal change if the line integrals are replaced by surface integrals taken over an arbitrary limited region, whether the functions $p_k(z)$ are polynomials or more general functions of class L^2. The functions $1, z, z^2, \cdots$ are linearly independent with respect to any positive continuous norm function, integration being taken over any finite region of the plane. Orthogonalization and normalization of this set yields normal orthogonal polynomials $p_k(z)$ of respective degrees $n = 0, 1, 2, \cdots$. Such polynomials were first studied systematically by Carleman [1922]. In particular the polynomials $1, z, z^2, \cdots$ themselves are orthogonal ($n(z) \equiv 1$) in every region $|z| < R$ or the corresponding closed region, as the reader may verify.

The polynomials $p_l(z)$, orthogonal as measured by a line or surface integral with a norm function, have immediate application to the cases in §§5.2–5.7 of approximation in the sense of least p-th powers with a norm function, with $p = 2$. In every such case, the polynomial $\pi_n(z)$ of degree n of best approximation is the sum of the first $n + 1$ terms of a series of the form

$$f(z) \sim a_0 p_0(z) + a_1 p_1(z) + \cdots; \tag{16}$$

the polynomials $p_k(z)$ are polynomials of respective degrees k normal and orthogonal on C or γ with respect to the norm function, and the coefficients a_k are given by (14) or by a similar line or surface integral taken over C or γ. In each case ($p = 2$) studied in §§5.2–5.7 where the given function $f(z)$ is analytic on the given closed point set, the relation (16) is an actual *equation*, and the right-hand member

converges to $f(z)$ on C or γ like a geometric series. If we multiply each member by its conjugate into $n(z)$ and integrate term by term (for definiteness let us choose a line integral over C), we have the equation of closure

$$\int_C n(z) \mid f(z) \mid^2 \mid dz \mid = \sum_{k=0}^{\infty} \mid a_k \mid^2 .$$

Let C be a closed limited region or more generally an arbitrary closed limited point set. If there exist polynomials $p_n(z)$ depending only on C such that an arbitrary function $f(z)$ analytic on C can be expanded uniformly on C in a series of the form

$$f(z) = a_0 p_0(z) + a_1 p_1(z) + \cdots , \qquad a_n \text{ constant},$$

then the set of polynomials $p_n(z)$ is said to *belong to* the point set C. An obvious illustration is the case that C is the closed interior of the unit circle and the $p_n(z)$ are the polynomials $1, z, z^2, \cdots$. The problem of the existence and determination of such polynomials for an arbitrary region was first proposed and solved (if the region is bounded by an analytic Jordan curve) by Faber [1903]. It is to be noted that in Chapter V we have determined sets of polynomials (depending on the arbitrary functions $n(z)$ or $n(w)$) belonging to various point sets C in the case (Theorem 3) that C is a region bounded by a rectifiable Jordan curve [Szegö, Smirnoff, Walsh]; in the case (Theorem 4) that C consists of several rectifiable Jordan arcs or Jordan regions bounded by rectifiable Jordan curves [Faber, Walsh and Russell]; in the case (Theorem 5) that C is an arbitrary closed limited region [Carleman, Walsh] or (Theorem 6) several such mutually exclusive regions [Walsh and Russell]; in the case (Theorem 7) that C is an arbitrary closed limited point set not a single point whose complement is simply connected or (Theorem 8) several such mutually exclusive point sets; and even (Theorem 11) in still more general cases. Indeed, our main purpose in introducing the measure of approximation used in Theorems 7 and 8 was to obtain polynomials belonging to very general point sets. The first solution of Faber's problem for the most general *Jordan* region is due to Fejér [1918]; see §7.6.

§6.6. Asymptotic properties of coefficients

The study of asymptotic properties of the coefficients in series of orthogonal polynomials [compare particularly Szegö, 1921] is now relatively simple.

THEOREM 7. *Let C be a rectifiable Jordan curve and let the function $f(z)$ be analytic interior to C_ρ but have a singularity on C_ρ, where ρ is finite or infinite. If the polynomials $p_n(z)$ of respective degrees n are normal and orthogonal on C with respect to the positive continuous norm function $n(z)$, then we have*

$$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} = 1/\rho , \qquad a_n = \int_C n(z) f(z) \overline{p_n(z)} \mid dz \mid . \tag{17}$$

Let $R < \rho$ be arbitrary. For z on C we have (§5.2, Theorem 3)

$$\left| f(z) - \sum_{k=0}^{n-1} a_k p_k(z) \right| \leqq M/R^{n-1},$$

where M is suitably chosen. From Schwarz's inequality (i.e., §5.7, inequality (30) for $\alpha = 1/2$) there follows

$$| a_n |^2 \leqq \int_C n(z) \left| f(z) - \sum_{k=0}^{n-1} a_k p_k(z) \right|^2 | dz | \cdot \int_C n(z) \mid p_n(z) \mid^2 \mid dz \mid,$$

$$a_n = \int_C n(z) \left[f(z) - \sum_{k=0}^{n-1} a_k p_k(z) \right] \overline{p_n(z)} \mid dz \mid, \qquad | a_n | \leqq M_1/R^n.$$

We find then that the left-hand member of (17) is not greater than $1/R$; the arbitrariness of $R < \rho$ yields

(18) $$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} \leqq 1/\rho.$$

In order to prove Theorem 7 it remains merely to show that

(19) $$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} = 1/\rho' < 1/\rho$$

is impossible. Let us assume (19) to hold; we shall reach a contradiction. Choose R and R_1 arbitrarily, $\rho < R < R_1 < \rho'$; from (19) follows the inequality

(20) $$| a_n | \leqq M'/R_1^n,$$

where M' is suitably chosen. Let us suppose $n(z) \geqq n_1 > 0$, so that we have

(21) $$\int_C n(z) \mid p_n(z) \mid^2 \mid dz \mid = 1, \qquad \int_C \mid p_n(z) \mid^2 \mid dz \mid \leqq 1/n_1.$$

By (20), (21), and the Lemma of §5.2 we find for z on or within C_R

$$| a_n p_n(z) | \leqq M'' (R/R_1)^n.$$

Then the formal expansion of $f(z)$ converges uniformly to $f(z)$ on and within C_R, $R > \rho$, which is impossible by the definition of ρ. The proof of Theorem 7 is complete.

The proof just given requires certain minor modifications to apply to the situation of §5.2, Theorem 4.

Corollary. *In the situation of* §5.2, *Theorem* 4, *let ρ be the largest number such that $f(z)$ is single-valued and analytic interior to C_ρ, where ρ is finite or infinite. If the polynomials $p_n(z)$ are normal and orthogonal on Γ with respect to the positive continuous norm function $n(z)$, then* (17) *is valid, where the integral is now taken over* Γ.

We prove (18) precisely as before, and prove that (19) is impossible by showing that it implies an impossibility. Inequality (20) follows from (19) as before, and we have

$$\int_{\Gamma} n(z) \mid p_n(z) \mid^2 \mid dz \mid = 1, \qquad \int_{\Gamma^{(j)}} n(z) \mid p_n(z) \mid^2 \mid dz \mid \leqq 1,$$

$$\int_{\Gamma^{(j)}} \mid p_n(z) \mid^2 \mid dz \mid \leqq 1/n_1.$$

Inequality (20) and the Lemma of §5.2 now yield for z on $[\Gamma^{(j)}]_{R_1/R}$ and hence for z on C

(22) $$\mid a_n p_n(z) \mid \leqq M_2/R^n, \qquad R > \rho.$$

Inequality (22) contradicts §4.7, Theorem 7, and the Corollary is established.

In particular, this Corollary includes the case that C consists of a finite number of segments of the axis of reals. The restriction on $n(z)$ is clearly made for convenience rather than ultimate generality.

Theorem 8. *Let C be a closed limited Jordan region and let the function $f(z)$ be analytic interior to C_ρ but have a singularity on C_ρ. If the polynomials $p_n(z)$ are normal and orthogonal on C as measured by a surface integral over C, with respect to the positive continuous norm function $n(z)$, then we have*

(23) $$\overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} = 1/\rho, \qquad a_n = \iint_C n(z)\, f(z)\, \overline{p_n(z)}\, dS.$$

The proof of (18) is valid under the present circumstances, with only the obvious modifications. We complete the proof of Theorem 8 by showing that (19) leads to a contradiction. Inequality (20) follows at once, $\rho < R < R_1 < \rho'$, and if we suppose $n(z) \geqq n_1 > 0$, we have

$$\iint_C n(z) \mid p_n(z) \mid^2 dS = 1, \qquad \iint_C \mid p_n(z) \mid^2 dS \leqq 1/n_1.$$

Let C' be an arbitrary closed region interior to C. By Lemma II of §5.3 we have for z on C'

$$\mid p_n(z) \mid \leqq M_3,$$

where M_3 is independent of n. For z on or within $C'_{R_1/R}$ we have by (20)

(24) $$\mid a_n p_n(z) \mid \leqq M_4/R^n.$$

By suitable choice of C', it follows (§2.1, Theorem 2) that (24) is valid for z on C, which is known to be impossible. The proof of Theorem 8 is complete.

We leave to the reader the proofs of the analogues of Theorems 7 and 8 for the situations of Chapter V, Theorems 6–10; no great difficulty is involved. We shall prove the following complement to the present Theorem 7 and leave also

to the reader the easy corresponding results for the related configurations treated in Chapter V.

THEOREM 9. *Let C be a rectifiable Jordan curve, let the polynomials $p_n(z)$ of respective degrees n be normal and orthogonal on the curve C with respect to the positive continuous weight function $n(z)$, and let the numbers a_n satisfy the relation $\overline{\lim}_{n\to\infty} | a_n |^{1/n} = 1/\rho < 1$. Then the function $f(z)$ defined on C by the equation*

$$f(z) = \sum_{n=0}^{\infty} a_n p_n(z) \tag{25}$$

is analytic interior to C_ρ but if not identically constant with $\rho = \infty$ has a singularity on C_ρ. Series (25) is the unique formal expansion on C of $f(z)$ in terms of the polynomials $p_n(z)$.

If $R_1 < \rho$ is arbitrary, we have for suitably chosen M

$$| a_n | \leqq M/R_1^n .$$

By (21) and the Lemma of §5.2 we have for z on or within C_R, $R < R_1$,

$$| a_n p_n(z) | \leqq M_0(R/R_1)^n ,$$

so the series (25) converges uniformly on any closed point set interior to C_{R_1}, hence uniformly on any closed point set interior to C_ρ. The function $f(z)$ is thus defined and analytic at every point interior to C_ρ.

We have already noted (§6.1) that in an expansion (25), uniformly valid on C, the coefficients are uniquely determined, and that the series itself is the formal expansion of the function represented. It follows now from Theorem 7 that $f(z)$ is analytic interior to C_ρ but has a singularity on C_ρ except of course in the trivial case that $f(z)$ is identically constant.

§6.7. Regions of convergence

There are two related topics concerning the expansions of Chapter V that we have not treated in detail: (a) asymptotic behavior of the approximating polynomials and of the orthogonal polynomials; (b) study of convergence or divergence of the developments exterior to C_ρ (in the usual notation). The methods of the present book involve especially degree of convergence of approximating sequences but in a relatively rough form, and we have not hitherto needed more delicate results under (a). The general question of the asymptotic behavior of even orthogonal polynomials has been treated in the literature only in isolated instances: by Szego [1921, 1921b, 1935] for orthogonality with an arbitrary weight function on an *analytic* Jordan curve or on a segment of the axis of reals; by Smirnoff [1928, 1932] for orthogonality ($n(z) \equiv 1$) on a more general but not the most general rectifiable Jordan curve; by Carleman [1922] for orthogonality ($n(z) \equiv 1$) in a region bounded by an analytic Jordan curve; even the

case of orthogonality on a single rectifiable Jordan curve with weight function unity has therefore not been treated completely.

In every case known, the normal orthogonal polynomials $p_n(z)$ satisfy the relation

$$\lim_{n\to\infty} | p_n(z) |^{1/n} = | \phi(z) |,$$

uniformly on any closed limited point set exterior to C containing no zero of $\phi'(z)$, where $\phi(z)$ is the usual mapping function for the exterior of the given set C. Whenever this condition is satisfied, the formal development of an arbitrary function $f(z)$ analytic on C converges interior to C_ρ (usual notation) and *diverges exterior to* C_ρ except perhaps at a zero of $\phi'(z)$; the latter remark (due to Szegö if C is an analytic Jordan curve with $n(z) \equiv 1$) is a consequence of Theorem 7 or its analogue and of §3.4, Theorem 5.

Further study of topic (a) would lead us too far afield from our present methods and from our major topic of approximation itself. Topic (b) has been studied to some extent in §4.8, Theorem 11, even for the expansions of Chapter V when $p \neq 2$. Another result now lies immediately at hand; this is stated for the situation of Theorem 7 but obviously extends directly to all of the situations of Chapter V, Theorems 4–10 with $p = 2$.

THEOREM 10. *Under the hypothesis of Theorem 7, the formal development*

$$f(z) = \sum_{n=0}^{\infty} a_n p_n(z)$$

can converge at every point of no region exterior to C_ρ.

In the proof of Theorem 10 we need to use the following theorem due to Osgood [1902], so we first prove

OSGOOD'S THEOREM. *If the sequence of functions $f_1(z)$, $f_2(z)$, $f_3(z)$, $\cdots$ all analytic in a region T converges in T, then the sequence converges uniformly in some subregion of T.*

It is sufficient, by Vitali's theorem, to prove that the sequence $f_n(z)$ is uniformly limited in some subregion of T. If the sequence $f_n(z)$ is not uniformly limited in T, there exists some n_1 and some point z_1 in T at which we have $|f_{n_1}(z)| > 1$. This inequality is valid at z_1, hence valid in some closed region T_1 interior to T. If the sequence $f_n(z)$ is not uniformly limited in T_1, there exists some $n_2 > n_1$ and some z_2 in T_1 at which we have $| f_{n_2}(z) | > 2$; this inequality is valid at z_2, hence valid in some closed region T_2 interior to T_1. If the theorem is not true, we have similarly $n_1 < n_2 < n_3 < \cdots$,

$$|f_{n_1}(z)| > 1, \qquad z \text{ in } T_1,$$
$$|f_{n_2}(z)| > 2, \qquad z \text{ in } T_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$|f_{n_k}(z)| > k, \qquad z \text{ in } T_k,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where the closed region T_k lies interior to T_{k-1}. The closed regions $T_1, T_2, \cdots$ have a common point z_0 (which may be defined as a limit point of the sequence z_n and which lies in T), and the sequence of functions $f_n(z)$ cannot converge in this point z_0. This contradiction completes the proof of Osgood's theorem.

We return now to the proof of Theorem 10, and assume the theorem false; we shall reach a contradiction. By virtue of Osgood's theorem we can assume the formal development of $f(z)$ to converge *uniformly* in some region T exterior to C_ρ. In particular we have the relation for all n

$$|a_n p_n(z)| \leqq M, \qquad z \text{ in } T,$$

where M (independent of n and z) is suitably chosen. Let z_0 (necessarily exterior to C_ρ) be a point interior to T, and let us set $|\phi(z_0)| = \sigma$, so that z_0 lies on the curve C_σ, $\sigma > \rho$.

It follows from (17) that the new series

$$F(z) = \sum_{n=0}^{\infty} a_n(\rho/\sigma)^n p_n(z)$$

satisfies the condition

$$\overline{\lim_{n\to\infty}}\, [a_n(\rho/\sigma)^n]^{1/n} = 1/\sigma,$$

and therefore (Theorem 9) the function $F(z)$ is analytic interior to C_σ but has a singularity on C_σ. Nevertheless we have for all n

$$|a_n(\rho/\sigma)^n\, p_n(z)| \leqq M(\rho/\sigma)^n, \qquad z \text{ in } T,$$

so the formal development of $F(z)$ converges uniformly in T, in contradiction to §4.8, Theorem 11. Theorem 10 is completely proved.

§6.8. Polynomials orthogonal on several curves

The discussion of §§6.1–6.5, in the light of §5.2 Theorem 3, suggests that the *same* system of polynomials may result from the orthogonalization with respect to suitable norm functions of the set $1, z, z^2, \cdots$ on two different curves C and C', and hence that the coefficients in the formal expansion of a suitably chosen analytic function can be found by integration over either C or C'. One important illustration of this fact is the case discussed in §6.1: the functions $1, z, z^2, \cdots$ are mutually orthogonal ($n(z) \equiv 1$) on *every* circle $|z| = R$ whose center is the

origin. Another illustration [Szegö, 1919] is that the set $1, z - c, z(z - c), z^2(z - c), \cdots$ is orthogonal on every circle $|z| = R > |c|$ with respect to the positive norm function $z/(R^2 - \bar{c}z)(z - c)$. We add now a few other remarks relating to orthogonality on several curves.

THEOREM 11. *Let C and C' be two distinct rectifiable Jordan curves and let the same set of polynomials $p_n(z)$ be obtained by orthogonalizing the set $1, z, z^2, \cdots$ on C with respect to the positive continuous norm function $n(z)$ and on C' with respect to the positive continuous norm function $n'(z)$. Then either C' is a curve C_R or C is a curve C'_R.*

The two curves C and C' are not identical, so we can suppose at least one point of C' to lie exterior to C. Let $\rho > 1$ be the smallest number such that C_ρ contains within it the interior of C'; such a smallest ρ exists, as may be seen by inspection of the corresponding situation in the w-plane after mapping the exterior of C onto the exterior of γ: $|w| = 1$ in the usual manner. If C' does not coincide with C_ρ, and we assume now that it does not, there exists a point $z = \zeta$ on C_ρ but exterior to C'; we shall reach a contradiction. Indeed, if ζ' is a variable auxiliary point near ζ but exterior to C_ρ, the formal expansion on C of the function $1/(z - \zeta')$ in terms of the polynomials $p_n(z)$ is also uniformly valid on C' and hence the two formal expansions on C and C' respectively are identical (§6.1) To be sure, the present polynomials $p_n(z)$ have not been supposed normal on C or C', but that fact does not essentially alter the reasoning previously given relative to term-by-term integration over C or C'. The coefficients in the formal expansion of $f(z) \equiv 1/(z - \zeta)$ on C and C' respectively are found by integration on C and C', and may also be found as the limits as ζ' approaches ζ of the corresponding coefficients in the formal expansion of $1/(z - \zeta')$ on C and C'. Thus the formal expansion of $f(z) \equiv 1/(z - \zeta)$ on C is identical with the formal expansion of $f(z)$ on C'.

Let z' denote a point of C' which lies on C_ρ. The formal expansion defined on C of $f(z) \equiv 1/(z - \zeta)$ converges uniformly throughout no neighborhood of z', by §4.8, Theorem 11. This same formal expansion, considered as defined on C', does converge uniformly throughout a suitably chosen neighborhood of z', for z' is on C' and ζ is exterior to C'. This contradiction completes the proof.

For definiteness Theorem 11 is stated for the case that C and C' are rectifiable Jordan curves, but the discussion is valid if either is a Jordan arc, or indeed if C or C' satisfies the hypothesis of any of Theorems 3–10 of Chapter V and orthogonality is understood in the corresponding sense. The discussion is valid also under much broader non-trivial conditions, as the reader will notice.

As an illustration of polynomials orthogonal on several curves we prove [Walsh, 1934a]

THEOREM 12. *The polynomials $p_n(z)$ found by orthogonalization of the set $1, z, z^2, \cdots$ on the line segment C: $-1 \leqq z \leqq +1$ with respect to the norm function*

$(1 - z^2)^{-1/2}$ *are also orthogonal with respect to the norm function* $|1 - z^2|^{-1/2}$ *on all the corresponding curves* C_R, *which are ellipses with the common foci* -1 *and* $+1$.

We shall find it more convenient to transform our problem to the w-plane. The exterior of C: $-1 \leqq z \leqq +1$ is transformed onto the exterior of γ: $|w| = 1$ by the transformation

$$z = \tfrac{1}{2}(w + 1/w) \tag{26}$$

so that the points at infinity correspond to each other. For the sake of reference we write the equations

$$\begin{aligned} z &= \frac{1}{2}(w + w^{-1}), \qquad z^2 = \frac{1}{4}(w^2 + 2 + w^{-2}), \\ z^3 &= \frac{1}{8}(w^3 + 3w + 3w^{-1} + w^{-3}), \cdots . \end{aligned} \tag{27}$$

The functions

$$p_n(z) \equiv w^n + 1/w^n$$

are polynomials in z of respective degrees n, by (27). These functions are mutually orthogonal on every circle Γ: $|w| = R \geqq 1$ with norm function unity, for we have $(n \neq k)$

$$\int_\Gamma p_n(z)\, \overline{p_k(z)}\, |dw| = \int_\Gamma (w^n + w^{-n})(R^{2k}w^{-k} + R^{-2k}\, w^k)R\, dw/(iw) = 0\,.$$

It remains to study the norm function in the z-plane, and to identify the $p_n(z)$ with the polynomials of Tchebycheff described in Theorem 12 From (26) we have for $|w| = R$,

$$dz = \frac{1}{2}(1 - w^{-2})dw\,, \qquad \left|\frac{dw}{dz}\right| = \frac{2R}{|w - w^{-1}|} = \frac{R}{|1 - z^2|^{1/2}} \tag{28}$$

The circle $|w| = 1$ corresponds to the segment $-1 \leqq z \leqq +1$ (counted twice, or in the study of orthogonality, only once if we prefer, for the norm function is single-valued on the segment) We already have

$$\int_{-1}^{1} n(z)\, p_n(z)\, \overline{p_k(z)}\, |dz| = \int_\gamma n(z) \left|\frac{dz}{dw}\right| p_n(z)\overline{p_k(z)}\, |dw|\,, \qquad n(z)\left|\frac{dz}{dw}\right| \equiv 1\,,$$

so the corresponding norm function on the segment $-1 \leqq z \leqq +1$ is $(1 - z^2)^{-1/2}$. The norm function on an arbitrary ellipse C_R whose foci are $+1$ and -1 is $R|1 - z^2|^{-1/2}$, where the ellipse is represented by $|z - 1| + |z + 1| = R + 1/R$, $R > 1$. The norm function on any curve can be modified by any non-vanishing constant factor, so if we prefer we can still express the norm function on any ellipse C_R as $|1 - z^2|^{-1/2}$. The proof is complete.

The polynomials $1, z, z^2, \cdots$ are orthogonal on an arbitrary circle $|z| = R$, and hence are orthogonal in an arbitrary region $|z| \leqq \rho$ with respect to any positive continuous norm function $n(z)$ which is a function of R alone, $n(z) \equiv n(R)$, $z = Re^{i\phi}$:

$$\iint_{|z| \leqq \rho} n(z)\, z^k \bar{z}^n \, dS = \int_0^\rho n(R)\, dR \int_0^{2\pi} z^k \bar{z}^n \, R d\phi = 0\,, \qquad k \neq n\,.$$

An analogous fact clearly holds for the polynomials $p_n(z)$ of Theorem 12:

COROLLARY. *The polynomials of Theorem 12 are mutually orthogonal in the interior of every ellipse whose foci are $+1$ and -1 with respect to the norm function $|1 - z^2|^{-1}$, or with respect to any positive continuous norm function whose quotient by $|1 - z^2|^{-1}$ is constant on every ellipse whose foci are $+1$ and -1.*

Again it is more convenient to study the situation in the w-plane. In any region S: $1 \leqq |w| \leqq \rho$ the polynomials $p_n(z)$ are mutually orthogonal with respect to any positive continuous norm function $n_1(z)$ constant on every circle $|w| = R$:

$$\begin{aligned} (29) \qquad & \iint_S n_1(z)\, p_k(z)\, \overline{p_n(z)}\, dS \\ &= \int_1^\rho n_1(z)\, dR \int_{|w|=R} p_k(z)\, \overline{p_n(z)}\, |dw| = 0\,, \qquad k \neq n\,, \end{aligned}$$

where w and z are connected by (26). The first integral in (29) can be written

$$\iint_\Sigma n_1(z)\, p_k(z)\, \overline{p_n(z)} \left| \frac{dw}{dz} \right|^2 d\Sigma\,,$$

where Σ is the interior of the ellipse $|z - 1| + |z + 1| = \rho + 1/\rho$. It follows from (28) that $p_k(z)$ and $p_n(z)$ are orthogonal on Σ with respect to the norm function $n_1(z)\, R^2\, |1 - z^2|^{-1}$; the function $n_1(z)\, R^2$ is positive, continuous, and constant on every circle $|w| = R$ whenever the same is true of the function $n_1(z)$, and conversely. The corollary is established.

The general problem of determining all complete sets of polynomials simultaneously orthogonal on several curves is still unsolved. Szegö [1935] determines all such sets orthogonal on every curve of an entire family C_R; the curve C must be a line segment, an ellipse, or a circle. Szegö also gives an alternate proof of Theorem 11 in the case that C and C' are analytic, and some simplifications due to him occur in the present proof of Theorem 12.

§6.9. Functions of the second kind

We shall now set forth an application of developments of analytic functions in orthogonal polynomials. For the sake only of simplicity we limit ourselves to the case of a single rectifiable Jordan curve. The discussion is valid also, with

merely obvious changes, in the case of a single rectifiable Jordan arc, and indeed in all the situations of Chapter V, Theorems 3–15, with the suitable measures of approximation.

LEMMA. *Let C be an arbitrary rectifiable Jordan curve, let $n(z)$ be positive and continuous on C, and let $p_k(z)$ be the set of polynomials normal and orthogonal on C with respect to the weight function $n(z)$. Then the expansion*

$$\frac{1}{t-z} = \sum_{k=0}^{\infty} p_k(z) \int_C \frac{n(z)\,\overline{p_k(z)}\,|dz|}{t-z} \tag{30}$$

is valid uniformly for z on C and for t on C_R.

Equation (30) is simply the expansion of the function $1/(t-z)$ in terms of the polynomials $p_k(z)$, and the uniformity with respect to t is the only unproved statement. Let R_1 be arbitrary, $1 < R_1 < R$. It follows from either of the two methods of §4.5 (or by the method of Shen, §7.8) that functions $P_n(z, t)$ which are polynomials in z of degree n exist such that we have

$$\left|\frac{1}{t-z} - P_n(z, t)\right| \leq \frac{M}{R_1^n}, \qquad z \text{ on } C, \ t \text{ on } C_R,$$

where M is independent of n, t, and z. By the method of §6.6 we then have

$$\left|\int_C \frac{n(z)\,\overline{p_k(z)}\,|dz|}{t-z}\right| \leq \frac{M_1}{R_1^n}, \qquad t \text{ on } C_R,$$

where M_1 is independent of n and of t. The Lemma now follows from the Lemma of §5.2, as in the proof of (22).

Let us introduce the notation

$$q_k(t) = \int_C \frac{n(z)\,\overline{p_k(z)}\,|dz|}{t-z}, \quad t \text{ exterior to } C;$$

the functions $q_k(t)$ are called *functions of the second kind.* It is clear from this definition that $q_k(t)$ is analytic everywhere in the extended plane exterior to C, and vanishes at infinity. Equation (30) can be written in the form

$$\frac{1}{t-z} = \sum_{k=0}^{\infty} p_k(z)\, q_k(t), \tag{31}$$

valid uniformly for z on C and t on C_R. Because of this uniformity, equation (31) is also valid uniformly for z on or within C and for t on or exterior to any C_R, $R > 1$.

Let $f_1(z)$ be an arbitrary function analytic on and within C, hence also analytic on and within some C_R; more generally, it is sufficient if $f_1(z)$ is analytic interior to C_R, continuous on and within C_R. Then we can write for z on or within C

$$(32) \qquad f_1(z) = \frac{1}{2\pi i}\int_{C_R}\frac{f_1(t)\,dt}{t-z} = \frac{1}{2\pi i}\sum_{k=0}^{\infty} p_k(z)\int_{C_R} f_1(t)\,q_k(t)\,dt.$$

Equation (32) is valid uniformly for z on C, hence the coefficients of the $p_k(z)$ must be identical with the coefficients in the formal expansion of $f_1(z)$ hitherto considered:

$$\int_C n(z)\,f_1(z)\,\overline{p_k(z)}\,|\,dz\,| = \frac{1}{2\pi i}\int_{C_R} f_1(t)\,q_k(t)\,dt.$$

The integral in this right-hand member may be taken over any contour which contains in its interior the curve C but has on or within it no singularity of $f_1(z)$, for $q_k(t)$ is analytic exterior to C. The series in (32) converges to $f_1(z)$ uniformly on any closed set interior to C_R.

Let $f_2(t)$ be an arbitrary function analytic in the extended plane exterior to C, continuous in the corresponding closed region, and zero at infinity. Then we have

$$f_2(t) = \frac{-1}{2\pi i}\int_C \frac{f_2(z)\,dz}{t-z}, \qquad t \text{ exterior to } C,$$

where the integral is taken in the positive sense with respect to the exterior of C. From (31) we can write

$$f_2(t) = \frac{-1}{2\pi i}\sum_{k=0}^{\infty} q_k(t)\int_C f_2(z)\,p_k(z)\,dz,$$

and this equation is valid uniformly for t on any C_R, hence uniformly for t on any closed set exterior to C. The integral in this right-hand member can be taken over C or over any contour that contains C in its interior.

We are now in a position to prove [Walsh, 1935b]

THEOREM 13. *Let C be an arbitrary rectifiable Jordan curve, let $n(z)$ be positive and continuous on C, let $p_k(z)$ denote the set of polynomials of respective degrees k normal and orthogonal on C with respect to $n(z)$, and let $q_k(t)$ denote the corresponding functions of the second kind. Let the function $f(z)$ be analytic in the annular region D bounded by C and C_R, continuous in the corresponding closed region. Let us set*

$$(33) \qquad f(z) \equiv f_1(z) + f_2(z), \qquad z \text{ in } D,$$

where $f_1(z)$ is analytic throughout the interior of C_R and $f_2(z)$ is analytic in the extended plane throughout the exterior of C and vanishes at infinity; both $f_1(z)$ and $f_2(z)$ are continuous in the corresponding closed regions. Then we have

$$(34) \qquad f_1(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty} p_k(z)\int_{C_R} f_1(t)\,q_k(t)\,dt = \frac{1}{2\pi i}\sum_{k=0}^{\infty} p_k(z)\int_{\Gamma} f(t)\,q_k(t)\,dt,$$

for z interior to C_R, uniformly on any closed set interior to C_R;

(35) $$f_2(t) = \frac{-1}{2\pi i} \sum_{k=0}^{\infty} q_k(t) \int_C f_2(z)\, p_k(z)\, dz = \frac{-1}{2\pi i} \sum_{k=0}^{\infty} q_k(t) \int_\Gamma f(z)\, p_k(z)\, dz\,,$$

for t exterior to C, uniformly on any closed set exterior to C. Here Γ represents any contour interior to D containing C in its interior; the integral in (34) *is to be taken counterclockwise, the integral in* (35) *to be taken clockwise.*

Equation (33) is simply the division of $f(z)$ into its components (§1.7). The equation

$$\int_{C_R} f_1(t)\, q_k(t)\, dt = \int_{C_R} f(t)\, q_k(t)\, dt$$

follows from the equation

$$\int_{C_R} f_2(t)\, q_k(t)\, dt = 0\ ;$$

the function $f_2(t)q_k(t)$ is analytic at every point of the extended plane exterior to C_R, continuous in the corresponding closed region, and vanishes at infinity to at least the second order. Hence the second and third members of (34) are equal. Similarly, the equation

$$\int_C f_2(z)\, p_k(z)\, dz = \int_C f(z)\, p_k(z)\, dz$$

is a consequence of the equation

$$\int_C f_1(z)\, p_k(z)\, dz = 0\ ;$$

the function $f_1(z)p_k(z)$ is analytic interior to C, continuous in the corresponding closed region The second and third members of (35) are equal. In (34) and (35) the integrals over C_R may be also taken over Γ, the integrals over C may be taken over C_R or Γ

Expansion (34), valid uniformly on C, is known to be unique (§6.1). In particular we may set $f(z) \equiv f_1(z) \equiv p_n(z)$. This yields the set of equations

(36) $$\begin{aligned} &\int_\Gamma p_n(t)\, q_k(t)\, dt = 0\,, \qquad k \neq n\,, \\ &\int_\Gamma p_n(t)\, q_n(t)\, dt = 2\pi i\,. \end{aligned}$$

Here Γ may be taken as any contour which contains C in its interior. Equations (36) show that the expansion (35), or any similar expansion in terms of the $q_k(t)$ valid uniformly on Γ, must also be unique.

Of course the equations

$$\int_\Gamma p_n(t)\, p_k(t)\, dt = 0, \qquad \int_\Gamma q_n(t)\, q_k(t)\, dt = 0,$$

are likewise valid.

Equations (33), (34), and (35) yield a method for finding the components of $f(z)$ simply by using the values of $f(z)$ on Γ. Under certain circumstances (which deserve further study) the functions $q_k(t)$ can be defined by continuity even on C itself; in this case the integrals in (34) over Γ can be replaced by integrals over C; also in equations (36), we may replace Γ by C.

If C is a circle, equations (34) and (35) express the development of $f(z)$ into a Laurent series. If C is a line segment or an ellipse, the functions $q_k(z)$ have been studied and the present results for that case established by other methods by Neumann [1862], Heine [1878], and later writers.

When, as in Theorem 12, the polynomials $p_k(z)$ are mutually orthogonal with respect to the norm function $n(z)$ on a set of curves C (necessarily of form C'_R with C' fixed), *the development* (31) *is independent of* C. To be sure, the present polynomials $p_k(z)$ are assumed normal on C and hence change with C. Nevertheless, the series in (31) is unique on each curve C, so when the polynomials $p_k(z)$ are altered by various constant factors, the corresponding functions $q_k(z)$ are altered by the reciprocals of those constant factors.

The orthogonal properties of the polynomials $p_k(z)$ do not enter largely in the proof of Theorem 13, and that theorem can be correspondingly extended. Under still broader conditions, let the expansion

$$\frac{1}{t-z} = \sum_{k=0}^{\infty} \phi_k(t, z)$$

be valid uniformly for t on a given contour C_1 and for z on a given contour C_2 interior to C_1. Let the function $f_1(z)$ be analytic on and interior to C_1; we have

$$(37)\quad f_1(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f_1(t)\, dt}{t-z} = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_{C_1} f_1(t)\, \phi_k(t, z)\, dt, \quad \text{uniformly on } C_2.$$

Let the function $f_2(t)$ be analytic on and exterior to C_2 and zero at infinity; we have

$$(38)\quad f_2(t) = \frac{1}{2\pi i}\int_{C_2} \frac{f_2(z)\, dz}{z-t} = \frac{-1}{2\pi i}\sum_{k=0}^{\infty}\int_{C_2} f_2(z)\, \phi_k(t, z)\, dz, \quad \text{uniformly on } C_1.$$

If the given expansion of $1/(t-z)$ is valid uniformly for t on the given C_1 and for z on an arbitrary closed set interior to C_1, then (37) is valid uniformly on an arbitrary closed set interior to C_1; if the given expansion of $1/(t-z)$ is valid uniformly for z on the given C_2 and for t on an arbitrary closed set exterior to C_2, then (38) is valid uniformly on an arbitrary closed set exterior to C_2. Whether or not we make such an additional assumption on the given expansion of

$1/(t - z)$, if $f(z)$ is analytic in the closed annular region bounded by C_1 and C_2, then in that region we can split $f(z)$ into its components $f_1(z)$ and $f_2(z)$; these two functions can be expanded in the two series (37) and (38) valid as before; the function $f_1(t)$ in the second and third members of (37) can be replaced by $f(t)$, and the function $f_2(z)$ in the second and third members of (38) can be replaced by $f(z)$.

§6.10. Functions of class H_2

The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$) is said to belong to the class H_2 when and only when the series $\sum_{n=0}^{\infty} |a_n|^2$ converges. Such functions are of much importance, particularly in the study of approximation in the sense of least squares, so we shall devote some attention to them. The appropriateness of the notation H_2 is due to Hardy's study of the means which occur as in (39) below.

If $f(z)$ belongs to H_2, the numbers a_n are bounded, so the series for $f(z)$ converges at every point interior to C: $|z| = 1$, and $f(z)$ is analytic at every point interior to C.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an arbitrary series convergent for $|z| < 1$. The series is dominated on Γ_r: $|z| = r < 1$ by a convergent series $\sum M_n$; we may choose for instance M_n equal to $|a_n r_1^n|$, $r < r_1 < 1$. Then we have

$$(39) \qquad I_r = \int_{\Gamma_r} |f(z)|^2 \, |dz| = r \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = 2\pi r \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

The function I_r is clearly monotonically non-decreasing as r increases.

A necessary and sufficient condition that I_r (defined now for an arbitrary function $f(z)$ analytic interior to C) *be uniformly limited for all $r < 1$ is that $f(z)$ belong to H_2.* Indeed, if $f(z)$ belongs to H_2, the extreme right-hand member of (39) is term by term less than $2\pi \sum_{n=0}^{\infty} |a_n|^2$, a convergent series independent of r. In this case we even have by Abel's theorem for real power series

$$(40) \qquad \lim_{r \to 1} I_r = 2\pi \sum_{n=0}^{\infty} |a_n|^2.$$

Conversely, if I_r is uniformly limited for all $r < 1$, it follows that $f(z)$ belongs to H_2. If we assume the contrary, we reach a contradiction. The series $\sum_{n=0}^{\infty} |a_n|^2$ diverges; if N is arbitrary we have for suitably chosen ν

$$2\pi \sum_{n=0}^{\nu} |a_n|^2 > N.$$

For suitably chosen $\delta > 0$, the inequality $1 - \delta < r < 1$ implies

$$2\pi r \sum_{n=0}^{\nu} |a_n|^2 r^{2n} > N, \qquad I_r > N,$$

which contradicts our hypothesis.

The functions $1, z, z^2, \cdots$ are orthogonal on C and are normalized by multi-

plying each function by $(2\pi)^{-1/2}$. If $f(z)$ belongs to H_2, its Taylor development $\sum_{n=0}^{\infty} a_n z^n$ is a series on C in terms of the functions $1, z, z^2, \cdots$, which converges in the mean on C (§6.3) and hence converges in the mean to some function $f_1(z)$ of class L^2 on C. This function $f_1(z)$ is [Lebesgue] almost everywhere on C the derivative of its indefinite integral and hence [Fatou, 1906, or see Bieberbach, 1927] is the boundary value of $f(z)$, taken on almost everywhere on C by normal approach or for approach to C in a triangle with one vertex on C and otherwise interior to C. For a function of class H_2 we shall henceforth use the same notation for the function interior to C and this boundary value on C. The function $f(z)$ on C then satisfies the equations (by §6.3, Theorem 3)

$$(41) \qquad a_n = \frac{1}{2\pi} \int_C f(z)\, \bar{z}^n \,|\, dz \,| = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz\,, \qquad n \geqq 0\,,$$

and the equation (Corollary 2 to Theorem 3)

$$\int_C |\, f(z)\,|^2 \,|\, dz\,| = 2\pi \sum_{n=0}^{\infty} |\, a_n \,|^2\,;$$

compare (40). Moreover, the functions $1, z, z^2, \cdots$ are not merely orthogonal to each other on C but also orthogonal to all functions of the set $z^{-1}, z^{-2}, \cdots$, so we have (Corollary 1 to Theorem 3)

$$\int_C f(z)\, \bar{z}^{-n} \,|\, dz\,| = 0\,, \qquad \int_C f(z)\, z^{n-1}\, dz = 0\,, \qquad n > 0\,.$$

Cauchy's integral for the arbitrary function $f(z)$ of class H_2 is valid for all points z interior to C if the integral is taken over C. Indeed, the function $\psi_\nu(z) = \sum_{n=0}^{\nu} a_n z^n$ converges in the mean on C to the function $f(z)$, from which it follows (§5.8, Theorem 16) that $\psi_\nu(z)$ converges interior to C to the function

$$\frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{t - z}\,, \qquad z \text{ interior to } C\,;$$

then this last function coincides with $f(z) \equiv \sum_{n=0}^{\infty} a_n z^n$ interior to C. It follows also by the method used in §5.8, Theorem 16, that we have

$$(42) \qquad \frac{1}{2\pi i} \int_C \frac{\psi_\nu(t)\, dt}{t - z} \equiv 0\,, \qquad \frac{1}{2\pi i} \int_C \frac{f(t)\, dt}{t - z} \equiv 0\,, \qquad z \text{ exterior to } C\,.$$

The results just established are due primarily to F. and M. Riesz [1920]. Let us state for reference

THEOREM 14. *If $f(z)$ is a function of class H_2, the development*

$$(43) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n\,, \qquad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz\,, \qquad |\, z\,| < 1\,,$$

found by formal expansion on C: $|z| = 1$ is identical with the expansion of $f(z)$ found by interpolation to $f(z)$ in the origin (§3.2). The sum of the first $n + 1$ terms of (43) is the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least squares.

The following relations hold:

$$(44) \qquad f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t - z}, \qquad |z| < 1; \qquad \int_C \frac{f(t)\,dt}{t - z} \equiv 0, \qquad |z| > 1,$$

$$(45) \qquad \int_C f(z)\, z^n\, dz = 0, \qquad n = 0, 1, 2, \cdots.$$

Theorem 14 is to be compared with §2.5, Theorem 10.

The class H_2 of functions on C is precisely the closed extension (§6.4) on C of the set $1, z, z^2, \cdots$. In particular (§6.4, Corollary 2 to Theorem 5), the limit $f(z)$ in the mean on C of a sequence $f_n(z)$ of functions of class H_2 is also of class H_2, and (§5.8, Theorem 16) the corresponding sequence of analytic functions $f_n(z)$ interior to C converges uniformly to the corresponding analytic function $f(z)$ on any closed set interior to C.

§6.11. Polynomials in z and $1/z$

Let $F(z)$ of class L^2 be given merely on the circumference C: $|z| = 1$. What can be said of the formal expansion of $F(z)$ on C? Let us prove

THEOREM 15. *Let $F(z)$ be of class L^2 on C: $|z| = 1$. The formal expansion*

$$(46) \qquad \sum_{n=0}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi}\int_C F(z)\, \bar{z}^n\, |dz| = \frac{1}{2\pi i}\int_C \frac{F(z)}{z^{n+1}}\, dz, \qquad |z| < 1,$$

represents some function $f(z)$ of class H_2 interior to C and converges to $f(z)$ in the mean on C. The relations

$$(47) \qquad \int_C \frac{F(z)}{z^{n+1}}\, dz = \int_C \frac{f(z)}{z^{n+1}}\, dz, \qquad n = 0, 1, 2, \cdots,$$

$$(48) \qquad f(z) = \frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t - z}, \qquad |z| < 1,$$

are valid. A necessary and sufficient condition that $f(z)$ should vanish identically is

$$f^{(n)}(0) = 0 \qquad \text{or} \qquad \int_C \frac{F(z)}{z^{n+1}}\, dz = 0, \qquad n = 0, 1, 2, \cdots.$$

There is no function $f_1(z)$ of class H_2 differing from $f(z)$ on C on a set of positive measure such that we have

$$\int_C |F(z) - f_1(z)|^2\, |dz| \leqq \int_C |F(z) - f(z)|^2\, |dz|.$$

The fact that (46) represents a function of class H_2 follows from Theorem 1, Corollary 2, and the relations (47) simply express the fact (Theorem 3) that the coefficients a_n in terms of the orthogonal set $1, z, z^2, \cdots$ can be found also from $f(z)$ by integration over C. Equation (48) is a consequence of (44) and (47), if we write

$$\frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t-z} = \frac{1}{2\pi i}\int_C F(t)\left[\frac{1}{t}+\frac{z}{t^2}+\frac{z^2}{t^3}+\cdots\right]dt, \qquad |z|<1;$$

it is permissible to integrate term by term; we note incidentally that (47) follows from (48). The validity of Cauchy's integral formula (48) implies the validity of the equations found from it formally by differentiation with respect to z. On C class H_2 is the closed extension of the normal orthogonal set $(2\pi)^{-1/2}, (2\pi)^{-1/2}z, (2\pi)^{-1/2}z^2, \cdots$. The last part of Theorem 15 now follows from Theorem 6.

Let us denote by G_2 the class of all functions of the form $g(z) \equiv \sum_{n=1}^{\infty} a_{-n}z^{-n}$, $|z|>1$, where the series $\sum_{n=1}^{\infty}|a_{-n}|^2$ converges. Such a function is analytic at every point of the extended plane exterior to C, vanishes at infinity, and possesses properties entirely analogous to properties of functions of class H_2. In particular $g(1/z)$ is a function of class H_2, so boundary values of $g(1/z)$, hence of $g(z)$, exist almost everywhere on C and we have for those boundary values

$$a_{-n} = \frac{1}{2\pi}\int_C g(z)\bar{z}^{-n}\,|dz| = \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{-n+1}}\,dz, \qquad n>0;$$

$$\int_C g(z)\bar{z}^{n}\,|dz| = \int_C \frac{g(z)}{z^{n+1}}\,dz = 0, \qquad n \geqq 0;$$

$$g(z) = \frac{-1}{2\pi i}\int_C \frac{g(t)\,dt}{t-z}, \qquad |z|>1; \qquad \int_C \frac{g(t)\,dt}{t-z} = 0, \qquad |z|<1.$$

For the sake of uniformity, we have expressed these equations so that all integrals are to be taken in the counterclockwise sense on C; similarly with (49) below.

The analogue of Theorem 15 for functions of class G_2 presents no difficulty; when $F(z)$ is given of class L^2 on C the function $g(z)$ is of class G_2:

$$g(z) = \sum_{n=1}^{\infty} a_{-n}z^{-n}, \qquad a_{-n} = \frac{1}{2\pi i}\int_C \frac{F(z)}{z^{-n+1}}\,dz, \qquad |z|>1,$$

$$\int_C \frac{F(z)}{z^{-n+1}}\,dz = \int_C \frac{g(z)}{z^{-n+1}}\,dz, \qquad n>0,$$

$$(49) \qquad g(z) = \frac{-1}{2\pi i}\int_C \frac{F(t)}{t-z}\,dt, \qquad |z|>1.$$

There is no function $g_1(z)$ of class G_2 differing from $g(z)$ on a set of positive measure on C such that we have

$$\int_C |F(z)-g_1(z)|^2\,|dz| \leqq \int_C |F(z)-g(z)|^2\,|dz|.$$

By the method used in Corollary 1 to Theorem 3 it follows that any two functions $f(z)$ and $f_1(z)$ of class H_2 satisfy the relation

$$\int_C f(z)\, f_1(z)\, dz = 0\,.$$

Any two functions $g(z)$ and $g_1(z)$ of class G_2 satisfy the relation

$$\int_C g(z)\, g_1(z)\, dz = 0\,.$$

Any function $g(z)$ of class G_2 is orthogonal on C to all the functions $1, z, z^2, \cdots$, and is therefore (Theorem 3, Corollary 1) orthogonal on C to every function $f(z)$ of class H_2:

$$\int_C f(z)\,\overline{g(z)}\,|\,dz\,| = \int_C \overline{f(z)}\,g(z)\,|\,dz\,| = 0$$

If $f(z)$ is of class H_2 on C, and if $f(0) = 0$, then $\overline{f(z)}$ is of class G_2 on C, for on C we have $\bar{z} = 1/z$. Similarly, if $g(z)$ is of class G_2 on C, then $\overline{g(z)}$ is of class H_2 on C and the function interior to C whose boundary values are $\overline{g(z)}$ vanishes at the origin.

We shall now prove

THEOREM 16. *Let $F(z)$ be of class L^2 on C: $|\,z\,| = 1$. The formal expansion of $F(z)$ on C in terms of the set $1, z, z^2, \cdots$ represents for $|\,z\,| < 1$ a function $f(z)$ of class H_2, and the formal expansion of $F(z)$ on C in terms of the set $z^{-1}, z^{-2}, \cdots$ represents for $|\,z\,| > 1$ a function $g(z)$ of class G_2. Equations* (48) *and* (49) *are valid. Almost everywhere on C we have $F(z) = f(z) + g(z)$.*

The orthogonal set of functions $\cdots, z^{-2}, z^{-1}, 1, z, z^2, \cdots$ (normalized by multiplication with $(2\pi)^{-1/2}$) is closed on C with respect to the class L^2, by Theorem 4, for the Fourier set of functions $1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cdots$ is closed* on C with respect to the class L^2 and (§2.4) any linear combination of functions of the one orthogonal set is a linear combination of functions of the other orthogonal set. The formal expansion of an arbitrary function $F(z)$ of L^2 on C in terms of the set $\cdots, z^{-2}, z^{-1}, 1, z, z^2, \cdots$ converges in the mean on C to the function $F(z)$. But (Theorem 15) the expansion of $F(z)$ in terms of the set $1, z, z^2, \cdots$ converges in the mean on C to a function $f(z)$ of class H_2, and the expansion in terms of the set $z^{-1}, z^{-2}, \cdots$ converges in the mean on C to a function $g(z)$ of class G_2. The sum of these latter two expansions converges in the mean to $f(z) + g(z)$, which is then identical with $F(z)$ almost everywhere on C.

* We indicate rapidly a proof of this fact. It follows from the definition of an improper Lebesgue integral that an arbitrary function of L^2 can be approximated in the mean on C as closely as desired by a limited measurable function. The latter can be approximated in the mean as closely as desired by a function which assumes only a finite number of distinct values (by the definition of integral), hence (by the definition of measurability) by a function which is piecewise continuous. The latter function can be approximated in the mean by a continuous function, and therefore (by Weierstrass's theorem, §2.4) by a trigonometric polynomial.

In case $F(z)$ is analytic on C, the sum of the two expansions of $F(z)$ on C is of course the Laurent development of $F(z)$ in an annular region containing C in its interior. The two functions $f(z)$ and $g(z)$ are *components* of $F(z)$ in the sense of §1.7. Theorem 16 can be interpreted as a generalization of Theorem 13 in the situation that C is a circle with $n(z) \equiv 1$.

Our former results imply the

COROLLARY. *A necessary and sufficient condition that $F(z)$ and $f(z)$ be equal almost everywhere on C is $g(z) \equiv 0$ or (the equivalent)*

$$\int_C \frac{F(z)}{z^{-n+1}}\, dz = 0, \qquad n > 0.$$

A necessary and sufficient condition that $F(z)$ and $g(z)$ be equal almost everywhere on C is $f(z) \equiv 0$ or (the equivalent)

$$\int_C \frac{F(z)}{z^{n+1}}\, dz = 0, \qquad n \geqq 0.$$

§6.12. An extremal problem, line integrals

The extremal problem whose solution is indicated by Theorem 15 admits a generalization, whose solution [Walsh, 1933b] can also be obtained by Theorem 15.

THEOREM 17. *Let $F(z)$ be of class L^2 on C: $|z| = 1$. Let $n(z)$ be positive and continuous on C. The essentially unique function $f(z)$ of class H_2 such that*

$$\int_C n(z)\, | F(z) - f(z) |^2\, | dz | \tag{50}$$

is least is given by

$$f(z) = \frac{1}{2\pi i N(z)} \int_C \frac{N(t) F(t)\, dt}{t - z}, \qquad |z| < 1, \tag{51}$$

where $N(z)$ is analytic interior to C and has boundary values such that $| N^2(z) | = n(z)$ almost everywhere on C.

Let $g(x, y)$ be harmonic interior to C, continuous on and within C, equal to $\frac{1}{2} \log n(z)$ on C, and let $h(x, y)$ be a function conjugate to $g(x, y)$ interior to C. The functions $N(z) \equiv e^{g+ih}$ and $1/N(z)$ are analytic and uniformly limited interior to C, hence [Fatou, 1906] have boundary values $N(z)$ and $1/N(z)$ almost everywhere on C. The function $N(z)$ on C is measurable and uniformly limited, and we have $| N^2(z) | = n(z)$ almost everywhere on C.

The integral (50) can now be written

$$\int_C | N(z) F(z) - N(z) f(z) |^2\, | dz |. \tag{52}$$

The problem of minimizing the integral (50) has now been reduced to the problem of minimizing the integral (52) whose norm function is unity. The function $N(z)F(z)$ is of class L^2 on C. The unknown function $N(z)f(z)$ is of class H_2, for [compare (39)] if $f(z)$ is of class H_2, so also is $N(z)f(z)$, and conversely.

If we now apply the last part of Theorem 15 to the problem of minimizing (52), we see that the minimizing function $f(z)$ is given by (51), and the proof is complete. The extremal property defines $f(z)$ on C uniquely, except at points of a set of measure zero.

A problem related to that of Theorem 17 can be solved by a similar method:

THEOREM 18. *Let $F(z)$ be of class L^2 on C: $|z| = 1$, and let $n(z)$ be positive and continuous on C. Let auxiliary conditions interior to C of the form*

$$f(\beta_k) = \gamma_k, \qquad |\beta_k| < 1, \qquad k = 1, 2, \cdots, \nu, \tag{53}$$

be prescribed. The essentially unique function $f(z)$ of class H_2 which satisfies these auxiliary conditions and which minimizes the integral (50) *is given by*

$$f(z) = p(z) + \frac{\Pi(z)}{2\pi i\, N(z)} \int_C N(t)\, \frac{F(t) - p(t)}{\Pi(t)(t - z)}\, dt, \qquad |z| < 1, \tag{54}$$

where $N(z)$ is defined as before, where $p(z)$ is a polynomial which satisfies the auxiliary conditions (53), *and where $\Pi(z)$ is defined by*

$$\Pi(z) = \prod_{k=1}^{\nu} \left(\frac{z - \beta_k}{1 - \bar{\beta}_k z}\right).$$

The points β_k need not be distinct, in conformity with the usual convention (§3 1).

For definiteness let $p(z)$ be the polynomial of degree $\nu - 1$ which satisfies the auxiliary conditions (53); it is also satisfactory if $p(z)$ is any other function of class H_2 which satisfies the auxiliary conditions, and it may be verified directly that the final formula (54) is independent of the particular choice of the function $p(z)$. The integral (50) can be written in the form

$$\int_C \left| N(z)\, \frac{F(z) - p(z)}{\Pi(z)} - N(z)\, \frac{f(z) - p(z)}{\Pi(z)} \right|^2 |dz|, \tag{55}$$

for $\Pi(z)$ is of modulus unity on C.

The extremal problem of Theorem 18 is now equivalent to a new problem suggested by (55). Given the function

$$F_1(z) = N(z)\, \frac{F(z) - p(z)}{\Pi(z)},$$

which is of class L^2 on C, to find the function

$$f_1(z) = N(z)\, \frac{f(z) - p(z)}{\Pi(z)} \tag{56}$$

of class H_2 which minimizes (55). It is to be noticed that when $f(z)$ is of class H_2 and satisfies the conditions (53), then $f_1(z)$ defined by (56) is also of class H_2; when $f_1(z)$ is of class H_2, then $f(z)$ defined by (56) is also of class H_2 and satisfies the auxiliary conditions (53).

We can now apply the last part of Theorem 15 to the problem of minimizing the integral (55). We obtain (54) by solving (56) for the minimizing function $f(z)$.

The restriction on $n(z)$ can be somewhat lightened without altering the conclusion or method of proof in both Theorem 17 and Theorem 18.

It will be noticed too that the reasoning applies in certain cases if the number of conditions (53) is infinite. We choose $p(z)$ as an arbitrary function of class H_2, if such exists, satisfying the auxiliary conditions. The product $\Pi(z)$ is to be replaced by the Blaschke product (assumed convergent) of §10.1. The reasoning remains valid, and the minimizing function $f(z)$ of class H_2 is represented by (54); compare §10.7.

For simplicity let us consider the situation of Theorem 17. If it is desired not merely to obtain the formula (51) for $f(z)$ but also to obtain an expansion of $f(z)$ by expanding $F(z)$ formally, we can use the polynomials orthogonal on C with respect to the function $n(z)$. Or if we prefer, we may introduce the functions (no longer necessarily polynomials)

$$P_k(z) \equiv z^k/N(z), \qquad k = 0, 1, 2, \cdots,$$

each analytic and uniformly limited interior to C, and with boundary values almost everywhere on C. These functions are mutually orthogonal on C with respect to the norm function $n(z)$:

$$\int_C n(z)\, P_n(z)\, \overline{P_k(z)}\, |dz| = \int_C z^n \bar{z}^k\, |dz| = 0, \qquad k \neq n.$$

The class H_2 is the closed extension on C of the set $P_k(z)$, by Corollary 5 to Theorem 5, for each function $P_k(z)$ is of class H_2; reciprocally, if n and $\epsilon > 0$ are given, the inequality

$$\int_C \left| N(z)\, z^n - \sum_{k=0}^{m} A_k z^k \right|^2 |dz| < n_1\epsilon, \qquad n(z) > n_1 > 0,$$

is satisfied for suitably chosen $m, A_0, A_1, \cdots, A_m$, and implies

$$\int_C \left| z^n - \sum_{k=0}^{m} A_k P_k(z) \right|^2 |dz| < \epsilon,$$

so each z^n belongs to the closed extension of the set $P_k(z)$.

Similarly, the situation of Theorem 18 corresponds to the formal expansion of $F_1(z)$ in terms of the set z^k orthogonal on C, or if we prefer, to the formal expansion of $F(z) - p(z)$ in terms of the set $Q_k(z) \equiv \Pi(z)\, z^k/N(z)$, $k = 0, 1, 2, \cdots$, orthogonal on C with respect to the norm function $n(z)$. The closed extension

on C of the set $Q_k(z)$ is precisely the class of functions of H_2 which vanish in the points β_j. Each function $Q_k(z)$ belongs to H_2, so this closed extension belongs to H_2. If $\phi(z)$ is a function of class H_2 which vanishes in the points β_j, the function $\phi(z)/\Pi(z)$ is of class H_2 (compare the condition connected with (39)), belonging therefore to the closed extension of the set $P_k(z)$, so $\phi(z)$ belongs to the closed extension of the set $Q_k(z)$.

Theorems 17 and 18 extend to more general regions by the use of conformal mapping; compare §11.5. The sets of functions $P_k(z)$ and $Q_k(z)$ have their analogues in this more general situation.

§6.13. An extremal problem, surface integrals

The functions $1, z, z^2, \cdots$ orthogonal on the circumference $|z| = 1$ have a closed extension H_2 that we have studied in §§6.10–6.12. But these functions $1, z, z^2, \cdots$ are also orthogonal in the (open) region C': $|z| < 1$. We propose now to study their closed extension H_2' in C'.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an arbitrary function analytic interior to C'. If we introduce the notation γ_r: $|z| \leqq r < 1$, we have

$$\iint_{\gamma_r} |z|^{2k}\, dS = \pi r^{2k+2}/(k+1),$$

$$\iint_{\gamma_r} |f(z)|^2\, dS = \sum_{k=0}^{\infty} \pi r^{2k+2} |a_k|^2/(k+1),$$

so a necessary and sufficient condition that $f(z)$ should be of class L^2 on C' (compare §6.10) is the convergence of $\sum_{k=0}^{\infty} |a_k|^2/(k+1)$. If $f(z)$ is of class H_2' on C', the function $f(z)$ defined almost everywhere on C' must by definition be of class L^2 on C' and (§5.8, Corollary 1 to Theorem 17; method of proof of §6.3, Lemma) is equal almost everywhere to a function analytic interior to C'. Reciprocally, a function

$$(57)\qquad \begin{aligned} &f(z) \equiv a_0 + a_1 z + a_2 z^2 + \cdots, \qquad |z| < 1, \\ &\sum_{k=0}^{\infty} |a_k|^2/(k+1) \text{ convergent}, \end{aligned}$$

is of class H_2', for we have

$$a_k = \frac{k+1}{\pi} \iint_{C'} f(z)\, \bar{z}^k\, dS,$$

$$\iint_{C'} \left| f(z) - \sum_{k=0}^{n} a_k z^k \right|^2 dS = \pi \sum_{k=n+1}^{\infty} |a_k|^2/(k+1),$$

which approaches zero with $1/n$; the fact that a function (57) is of class H_2' follows also from §2.7, Theorem 14. We have proved

THEOREM 19. *The class H_2' which is the closed extension of the set $1, z, z^2, \cdots$ on C': $|z| < 1$ is essentially the set of functions of class L^2 on C' analytic in the interior points of C', or in other words is essentially the class of functions represented by* (57).

The analogue of the latter part of Theorem 15 is

THEOREM 20. *Let $F(z)$ be of class L^2 on C'. The essentially unique function $f(z)$ of class H_2' such that*

$$\iint_{C'} |F(z) - f(z)|^2 \, dS$$

is least is given by

$$f(z) \equiv \frac{1}{\pi} \iint_{C'} F(\zeta) \frac{dS}{(1 - \bar{\zeta}z)^2}, \qquad |z| < 1. \tag{58}$$

The formal development of $F(z)$ on C' in terms of the functions z^k is

$$\sum_{k=0}^{\infty} a_k z^k, \qquad a_k = \frac{k+1}{\pi} \iint_{C'} F(\zeta) \, \bar{\zeta}^k \, dS; \tag{59}$$

this series converges to $f(z)$ of class H_2' in the mean on C', hence (§5.8, Theorem 17) converges to $f(z)$ uniformly on any closed set interior to C'. Interior to C', the function represented by (59) is

$$f(z) \equiv \frac{1}{\pi} \iint_{C'} F(\zeta) \, [1 + 2\bar{\zeta}z + 3\bar{\zeta}^2 z^2 + \cdots] \, dS, \qquad |z| < 1,$$

for the series in square brackets converges uniformly for $|\zeta| \leqq 1$ when z is fixed. This equation for $f(z)$ can be rewritten in form (58).

Theorem 20 is due to Wirtinger [1932], by a quite different method.

By the method of proof of Theorem 17 the reader may prove

THEOREM 21. *Let $n(z)$ be the square of the modulus of a function $N(z)$ analytic, bounded, and bounded from zero on C': $|z| < 1$. Let the function $F(z)$ be of class L^2 on C'. The essentially unique function $f(z)$ of class H_2' such that*

$$\iint_{C'} n(z) \, |F(z) - f(z)|^2 \, dS \tag{60}$$

is least is given by

$$f(z) \equiv \frac{1}{\pi N(z)} \iint_{C'} \frac{F(\zeta) \, N(\zeta) \, dS}{(1 - \bar{\zeta}z)^2}, \qquad |z| < 1.$$

If auxiliary conditions (53) are prescribed in the situation of Theorem 21, the determination of $f(z)$ is more difficult. However, if all of the points β_i coincide

at the origin, we shall indicate that a solution of the problem lies at hand. In the case $n(z) \equiv 1$ let $p(z)$ denote the polynomial of degree $\nu - 1$ which satisfies the auxiliary conditions. The problem of determining $f(z)$ so as to minimize (60) is equivalent to that of determining the function $f_1(z)$ of class H_2' such that

$$\iint_{C'} | F(z) - p(z) - z^\nu f_1(z) |^2 \, dS$$

is least. The formal development of $F(z) - p(z)$ in terms of the functions $z^\nu, z^{\nu+1}, z^{\nu+2}, \cdots$ orthogonal on C' is

$$\sum_{k=\nu}^{\infty} a_k z^k, \qquad a_k = \frac{k+1}{\pi} \iint_{C'} [F(\zeta) - p(\zeta)] \, \bar{\zeta}^k \, dS,$$

whence as with (59)

$$z^\nu f_1(z) \equiv \frac{1}{\pi} \iint_{C'} [F(\zeta) - p(\zeta)] \frac{(\nu+1) \, \bar{\zeta}^\nu z^\nu - \nu \bar{\zeta}^{\nu+1} z^{\nu+1}}{(1 - \bar{\zeta} z)^2} \, dS;$$

the minimizing function $f(z)$ is $p(z) + z^\nu f_1(z)$. The introduction of a norm function as in Theorem 21 presents no difficulty.

Theorems 20 and 21 and the remark just made extend to more general regions by the use of conformal mapping; compare §11.4.

CHAPTER VII

INTERPOLATION BY POLYNOMIALS

§7.1. Interpolation in roots of unity

We studied in Chapter III the convergence of various sequences of polynomials found by interpolation. The points there chosen for interpolation are particularly simple, and the convergence properties of the sequences of polynomials are correspondingly simple. In the present chapter we shall study more complicated situations, where the distinct points of interpolation are no longer finite in number and where the boundaries of regions of convergence may no longer be lemniscates but may be more general curves.

Our fundamental problem is still the following: *Given a function $f(z)$ analytic on a closed point set C and a set of points*

$$
\begin{array}{ll}
& \beta_1^{(0)}, \\
& \beta_1^{(1)}, \beta_2^{(1)}, \\
(1) & \cdots\cdots\cdots, \\
& \beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}, \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{array}
$$

which either lie on C or have no limit point except on C; to study the convergence to $f(z)$ of the sequence of polynomials $p_n(z)$ of respective degrees n found by interpolation to $f(z)$ in the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$. The $n + 1$ points of this set naturally need not be all distinct (§3.1).

In studying this problem, there are various properties of the sequence (1) which it may be natural and suitable to assume: (a) the numbers $\beta_k^{(n)}$ may be given explicitly, for instance $\beta_k^{(n)} = 0$, or the $\beta_k^{(n)}$ are all $n + 1$ of the $(n + 1)$-st roots of unity; (b) asymptotic properties of the $\beta_k^{(n)}$ may be given directly, as in §3.5; (c) asymptotic properties of the $\beta_k^{(n)}$ may be given indirectly, for instance in such a form as

$$\lim_{n\to\infty} | (z - \beta_1^{(n)}) (z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)}) |^{1/(n+1)} = | \Phi(z) |,$$

where the function $\Phi(z)$ is given; (d) asymptotic inequalities may be given, for instance

$$| \Phi_0(z) | \leqq | (z - \beta_1^{(n)}) (z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)}) |^{1/(n+1)} \leqq | \Phi_1(z) |,$$

where $\Phi_0(z)$ and $\Phi_1(z)$ are given; (e) geometric conditions may be given on the $\beta_k^{(n)}$, such as requiring that the $\beta_k^{(n)}$ should all lie in some circle or other region, say a given subregion of C. Examples of (a) are to be found in Chapter III, and in Theorems 1 and 2 (points equally distributed) below. The situation of

(c) is important, as indicated in Theorems 2 and 3 below. All of these situations (a) – (c) are to be studied in more generality in Chapter VIII.

Let us consider first a case where the points of interpolation are given explicitly; this particular choice of points of interpolation was made for a very special function by Méray (see §3.2), and for an arbitrary analytic function was introduced by Runge [1904]; proof of convergence of the sequence $p_n(z)$ to the function $f(z)$ interior to C is due to Runge.

THEOREM 1. *Let the function $f(z)$ be analytic for $|z| < \rho > 1$ but have a singularity on the circle $|z| = \rho$; the case that ρ is infinite is not excluded. Let $p_n(z)$ be the polynomial of degree n which coincides with $f(z)$ in the $(n+1)$-st roots of unity. Then the sequence $p_n(z)$ converges maximally to $f(z)$ on C: $|z| = 1$.*

Moreover, if $P_n(z)$ is the polynomial of degree n which coincides with $f(z)$ in the origin, considered of multiplicity $n+1$, then we have

$$\lim_{n\to\infty} [p_n(z) - P_n(z)] = 0 \tag{2}$$

for $|z| < \rho^2$, uniformly for $|z| \leqq Z < \rho^2$.

The polynomial $P_n(z)$ is also the polynomial of degree n of best approximation to $f(z)$ on C in the sense of least squares.

We have (§3.1)

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z^{n+1} - 1) f(t)\, dt}{(t^{n+1} - 1)(t - z)}, \qquad z \text{ interior to } \Gamma,$$

where Γ is a circle $|z| = R$, $1 < R < \rho$. For z on or within C the right-hand member is in absolute value not greater than a suitable constant (i.e., independent of n and z) divided by $(R^{n+1} - 1)$, and hence is not greater than a constant divided by R^{n+1}. The first part of the theorem follows at once.

We have also

$$f(z) - P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{z^{n+1} f(t)\, dt}{t^{n+1}(t - z)}, \qquad z \text{ interior to } \Gamma,$$

$$p_n(z) - P_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^{n+1} - z^{n+1}) f(t)\, dt}{t^{n+1}(t^{n+1} - 1)(t - z)}, \qquad z \text{ interior to } \Gamma. \tag{3}$$

When the integrand in (3) is suitably defined for $t = z$, that integrand has no singularity in t for $t = z$, the right-hand member of (3) is analytic for all (finite) values of z, so (3) is valid for all values of z. In particular for $|z| = Z < \rho^2$, the right-hand member approaches zero uniformly provided the expression

$$\frac{R^{n+1} + Z^{n+1}}{R^{n+1}(R^{n+1} - 1)}$$

approaches zero, and this condition is satisfied if we have $Z < R^2$. Equation (2) is established.

It will be noted that equation (2) implies a number of results on the convergence and divergence of the sequence $p_n(z)$, such as the divergence for $R < |z| < R^2$, the analogue of Abel's theorem for the sequence $p_n(z)$, etc.

The number ρ^2 that appears in (2) is by no means accidental, for let us set $f(z) = 1/(z - \rho)$. We have

$$f(z) - p_n(z) = \frac{z^{n+1} - 1}{(\rho^{n+1} - 1)(z - \rho)}, \qquad f(z) - P_n(z) = \frac{z^{n+1}}{\rho^{n+1}(z - \rho)},$$

$$p_n(z) - P_n(z) = \frac{\rho^{n+1} - z^{n+1}}{\rho^{n+1}(\rho^{n+1} - 1)(z - \rho)}.$$

For the particular value $z = \rho^2$, the difference

$$p_n(z) - P_n(z) = 1/(\rho - \rho^2)$$

does not approach zero.

The entire conclusion of Theorem 1 (the proof requires only minor changes) is valid also if the polynomial $p_n(z)$ is defined from $f(z)$ not by interpolation in the $(n + 1)$-st roots of unity but by interpolation in the $(n + 1)$-st roots of an arbitrary number α_n, where $|\alpha_n| \leq 1$.

§7.2. A sufficient condition for maximal convergence

The first part of Theorem 1 is a consequence of a much more general theorem [Walsh, 1933c]:

THEOREM 2. *Let C be a closed limited point set whose complement K with respect to the extended plane is connected and regular. Let $w = \phi(z)$ map K onto the region $|w| > 1$ so that the points at infinity correspond to each other. Let the points* (1) *have no limit point exterior to C and satisfy the relation*

$$\lim_{n\to\infty} |(z - \beta_1^{(n)})(z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)})|^{1/(n+1)} = \Delta |\phi(z)| \tag{4}$$

uniformly on any closed limited point set interior to K, where Δ is the capacity (§4.4) *of C. Let the function $f(z)$ be single-valued and analytic on C. Then the sequence of polynomials $p_n(z)$ of respective degrees n found by interpolation to $f(z)$ in the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$, converges to $f(z)$ maximally on C.*

Let $f(z)$ be single-valued and analytic interior to C_ρ but not interior to any $C_{\rho'}$, $\rho' > \rho$. Let $R < \rho$ be arbitrary, and choose R_1 and R_2, $R < R_1 < \rho$, $1 < R_2 < R_1/R$. For sufficiently large n, the points $\beta_k^{(n)}$ lie interior to C_{R_1}, so we have

$$f(z) - p_n(z) = \frac{1}{2\pi i}\int_{C_{R_1}} \frac{\omega_n(z)\, f(t)\, dt}{\omega_n(t)\,(t - z)}, \qquad z \text{ interior to } C_{R_1}, \tag{5}$$

$$\omega_n(z) = (z - \beta_1^{(n)})(z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)}). \tag{6}$$

We have uniformly

$$\lim_{n\to\infty} | \omega_n(z) |^{1/(n+1)} = \Delta R_2, \qquad z \text{ on } C_{R_2},$$

$$\lim_{n\to\infty} | \omega_n(t) |^{1/(n+1)} = \Delta R_1, \qquad t \text{ on } C_{R_1},$$

from which we deduce

$$\overline{\lim_{n\to\infty}} [\max | f(z) - p_n(z) |, z \text{ on } C_{R_2}]^{1/(n+1)} \leq R_2/R_1 < 1/R;$$

this inequality holds similarly if we replace the condition that z should lie on C_{R_2} by the condition that z should lie on C. The corresponding superior limit is therefore not greater than $1/\rho$, and cannot (§4.7) be less than $1/\rho$. The proof is complete.

This last inequality holds even if the points $\beta_k^{(n)}$ are not defined for every n, but merely for some infinite sequence of indices n.

REMARK. If the points $\beta_k^{(n)}$ have no limit point in K and if (4) holds for every z in K without the requirement of uniformity, then (4) holds *uniformly* on any closed limited point set interior to K, for (§7.3, below) the functions in (4) form a normal family in any closed limited region in K.

If we allow the closed interior of C_R to take the rôle of C, the right-hand member of (4) remains unchanged, for Δ is multiplied by R and $\phi(z)$ is divided by R. The points $\beta_k^{(n)}$ have no limit point exterior to C_R, so the sequence $p_n(z)$ converges maximally on C_R whenever $f(z)$ is single-valued and analytic on and within C_R. This remark follows too by the general theory of maximal convergence.

It occurs not infrequently in practice that the points $\beta_k^{(n)}$ are not given directly and are not given even by means of the polynomials $\omega_n(z)$, but that polynomials of the form $a_n\omega_n(z)$ are given whose roots are chosen as the $\beta_k^{(n)}$.

COROLLARY 1. *Let C be a closed limited point set whose complement K is connected and regular. Let $w = \phi(z)$ map K onto the region $| w | > 1$ so that the points at infinity correspond to each other. Let the points $\beta_k^{(n)}$ be defined as the $n + 1$ roots of the polynomials $\theta_n(z)$ which have respectively $n + 1$ roots. Let us suppose*

$$\lim_{n\to\infty} | \theta_n(z) |^{1/(n+1)} = \sigma | \phi(z) |, \qquad \sigma \neq 0, \tag{7}$$

uniformly on any closed limited point set interior to K. Let the function $f(z)$ be analytic on C. Then the sequence of polynomials $p_n(z)$ of respective degrees n found by interpolation to $f(z)$ in the points $\beta_k^{(n)}$ converges maximally to $f(z)$ on C.

It follows from the uniformity of (7) that the points $\beta_k^{(n)}$ have no limit point in K. The derivation of equation (5) is now valid if $\omega_n(z)$ and $\omega_n(t)$ are replaced by $\theta_n(z)$ and $\theta_n(t)$, so the proof goes through as before.

The first part of Theorem 1 is a direct consequence of Theorem 2, as are also the results of §§3.4 and 3.5.

COROLLARY 2. *Under the hypothesis of Corollary 1 on C, K, and the polynomials $\theta_n(z)$, and if K is simply connected, it is sufficient for the conclusion if the points $\beta_k^{(n)}$ are chosen not as the roots of the $\theta_n(z)$ but as the roots of the derivatives $\theta'_{n+1}(z)$, or as the roots of the second derivatives $\theta''_{n+2}(z)$, etc. Indeed, condition* (7) *valid uniformly on any closed point set interior to the simply connected region K implies the condition*

$$\lim_{n\to\infty} |\, \theta'_n(z)\,|^{1/n} = \sigma\, |\, \phi(z)\,|, \qquad \sigma \neq 0, \tag{8}$$

uniformly on any closed limited point set interior to K.

Modify $\theta_n(z)$ if necessary multiplying by a constant of modulus unity so that the coefficient of the highest power of z becomes positive. Let both $[\theta_n(z)]^{1/(n+1)}$ and $\phi(z)$ be so chosen that their derivatives at infinity are positive. Then (7) implies

$$\lim_{n\to\infty} [\theta_n(z)]^{1/(n+1)} = \sigma\, \phi(z)$$

uniformly on any closed limited point set S interior to K, as we shall prove. By (7) the functions $[\theta_n(z)]^{1/(n+1)}$ are uniformly limited on every S, hence form a normal family in every region S, hence also in K (with the point at infinity deleted). Every limit function of the family in K has the same modulus, by (7), and has a positive or zero derivative at infinity. In fact uniform convergence of a subsequence in every S implies uniform convergence of the quotient by z of the functions of the subsequence on a large circumference and therefore in a neighborhood of the point at infinity, hence implies approach at infinity of the derivatives of the functions of the subsequence to the derivative of the limit function. The quotient by $\sigma\phi(z)$ of any limit function of the family has a constant modulus in K, hence (Principle of Maximum) is a constant of modulus unity, hence (by the derivatives at infinity) has the value unity. That is to say, the sequence of functions $[\theta_n(z)]^{1/(n+1)}$ forms a normal family in every S (compare also §7.3 below) and converges at every finite point of K to the function $\sigma\phi(z)$, so converges uniformly in every S.

The following equations are now uniformly valid on any closed limited point set S interior to K, when we make use of the fact that $\phi(z)$ and $\phi'(z)$ are different from zero on every such point set:

$$\lim_{n\to\infty} \frac{[\theta_n(z)]^{1/(n+1)}}{\phi(z)} = \sigma,$$

$$\lim_{n\to\infty} \frac{\phi(z)\,[\theta_n(z)]^{-1+1/(n+1)}\,\theta'_n(z) - (n+1)\,[\theta_n(z)]^{1/(n+1)}\,\phi'(z)}{(n+1)\,[\phi(z)]^2} = 0,$$

$$\lim_{n\to\infty} \frac{\phi(z)\,\theta'_n(z)}{(n+1)\,\theta_n(z)} - \phi'(z) = 0, \qquad \lim_{n\to\infty} \frac{[\theta'_n(z)]^{1/n}}{(n+1)^{1/n}\,[\theta_n(z)]^{1/n}} = 1,$$

$$\lim_{n\to\infty} [\theta'_n(z)]^{1/n} = \sigma\, \phi(z),$$

as we were to prove.

Corollary 2 is not true if the requirement that K be simply connected is omitted, for if K is multiply connected (we fasten our attention on a particular branch of $[\theta_n(z)]^{1/(n+1)}$), there are points of K at which $\phi'(z)$ vanishes, so the derivation of (8) is not valid at such points. Indeed, it is clear from (7) that

$$\lim_{n \to \infty} \theta_n'(z_0)\,[\theta_n(z_0)]^{-n/(n+1)}/(n+1) = 0$$

if z_0 is a point of K such that $\phi'(z_0) = 0$, and hence (by Hurwitz's theorem) that in an arbitrary neighborhood of z_0 and for n sufficiently large every function $\theta_n'(z)$ has a zero. Interpolation to the given function $f(z)$ in the roots of $\theta_n'(z)$, as contemplated in Corollary 2, may not be possible because the function $f(z)$ may not be defined in the neighborhood of such a point z_0. However, the conclusion of Corollary 2 regarding interpolation is valid even if K is multiply connected, provided the polynomial $p_n(z)$ of interpolation is required to coincide with $f(z)$ at points $\beta_k^{(n)}$ in the neighborhood of a point z_0 (of the kind just considered) at which $f(z)$ is analytic; if $f(z)$ is not defined or not analytic at such a point z_0, we may assign to it throughout the neighborhood of z_0 the values of any constant or other analytic function. As in equation (5) we take the integral over C_{R_1} and also over a small circle exterior to C_{R_1} about each point z_0 exterior to C_{R_1}; we may assume no z_0 to lie on C_{R_1} and may assume now that $f(z)$ is analytic on and within this circle. The points $\beta_k^{(n)}$ have no limit point interior to K other than such a point z_0. The relation

$$\varliminf_{n \to \infty} |\,\omega_n(t)\,|^{1/(n+1)} \geqq \sigma R_1\,, \qquad \omega_n(z) \equiv \theta_n'(z)\,,$$

is valid uniformly on the contours of integration. Maximal convergence on C follows as before

In a similar way, the roots of the higher derivatives of the $\theta_n(z)$ may be used as points of interpolation; maximal convergence on C is a consequence.

The polynomials $p_n(z)$ that appear in Theorem 2 form a sequence of polynomials of interpolation, and clearly need not be the partial sums of a series of interpolation; for instance the sequence $p_n(z)$ of Theorem 1 is not derived from a series of interpolation. However, under the hypothesis of Theorem 2 we can derive a series of interpolation merely by choosing as points of interpolation β_ν all the points (1) taken in order, row by row. We still have maximal convergence to $f(z)$ on C:

COROLLARY 3. *Under the hypothesis of Theorem 2 let us denote by $\beta_1, \beta_2, \cdots$ the sequence of points* (1), *taken in order in each row and with the rows taken in order. Then we have*

$$\lim_{\nu \to \infty} |\,(z-\beta_1)\,(z-\beta_2)\cdots(z-\beta_\nu)\,|^{1/\nu} = \Delta\,|\,\phi(z)\,| \tag{9}$$

uniformly on any closed limited point set interior to K containing no point β_k.

We notice first the validity of the equation

$$(10)\quad \lim_{n\to\infty} | (z - \beta_1^{(0)}) (z - \beta_1^{(1)}) (z - \beta_2^{(1)}) (z - \beta_1^{(2)}) \cdots (z - \beta_{n+1}^{(n)}) |^{2/(n+1)(n+2)} = \Delta | \phi(z) |$$

uniformly on any closed limited point set S interior to K containing no point $\beta_k^{(n)}$. Indeed, we need merely consider the sequence

$$(11)\qquad \frac{\log | z - \beta_1^{(0)} |}{1}, \frac{\log | (z - \beta_1^{(1)}) (z - \beta_2^{(1)}) |}{2}, \cdots, \frac{\log | (z - \beta_1^{(n)}) (z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)}) |}{n+1}, \cdots,$$

whose limit is $\log \Delta | \phi(z) |$. Let us replace (11) by the new sequence in which the first term of (11) is unchanged, in which the second term of (11) then appears twice, in which the third term of (11) then appears three times, and so on. If the new sequence is summed by the method of Cesàro, which does not alter the uniform limit of the sequence (compare §3.5), equation (10) follows.

If we write the sequence corresponding to (9) in logarithmic form and make use of (10), we see that in order to prove (9) it is now sufficient to prove

$$(12)\qquad \lim_{n\to\infty} \frac{\log | (z - \beta_1^{(n)}) (z - \beta_2^{(n)}) \cdots (z - \beta_\mu^{(n)}) |}{\mu + n(n+1)/2} = 0, \qquad 0 < \mu \leqq n,$$

uniformly on S; to be sure, another term appears involving $(z - \beta_1^{(0)}) \cdots (z - \beta_n^{(n-1)})$, but that term can be treated by the same method and presents no difficulty. For z on S, the factors $| z - \beta_\nu |$ are uniformly limited and uniformly bounded from zero. Then the μ quantities $\log | z - \beta_k^{(n)} |$ in (12) are each uniformly limited and in number not greater than n, so (12) follows at once.

Corollary 3 enables us (compare §3.2) under suitable circumstances to replace any *sequence* of polynomials of interpolation by a corresponding *series* of interpolation, and the series of interpolation converges maximally to $f(z)$ on C whenever $f(z)$ is analytic on C and the points β_k all belong to C. In particular, if C is the unit circle we may choose the β_k as all the roots of unity taken in order of increasing index, for instance $1, \omega_2, \omega_2^2, \omega_3, \omega_3^2, \omega_3^3, \omega_4, \cdots$, where ω_n is a primitive n-th root of unity. A disadvantage of the new series of interpolation as compared with the sequence of interpolation is that if the same point $\beta_k^{(n)}$ occurs for several values of n, the value of the function and *of certain of its derivatives* at $\beta_k^{(n)}$ must be known in expanding the function in the series of interpolation.

Corollary 3, together with Theorem 2, does not in its present form imply that the series of interpolation to $f(z)$ (analytic on C) in the points β_ν converges maximally to $f(z)$ on C unless all the β_ν lie on C; for although the points β_ν have no limit point exterior to C, the set of points of interpolation

$$\begin{array}{l}\beta_1\,,\\ \beta_1,\,\beta_2\,,\\ \beta_1,\,\beta_2,\,\beta_3\,,\\ \cdots\cdots\cdots\cdots\,,\end{array}$$

used in defining the partial sums of the series of interpolation may have limit points exterior to C. Indeed, the analytic function $f(z)$ may fail to be defined at some of the points β_ν; in this case arbitrary fixed values may be assigned at such points. There are now two convenient methods of proving maximal convergence on C: (a) the transformation used in §3.5 may be introduced here, to avoid the occurrence in (5) of points β_ν not interior to C_{R_1}; (b) instead of taking the integral in (5) around C_{R_1}, the integral may be taken around C_{R_1} and around small circles exterior to each other and to C_{R_1} about the points β_ν which lie exterior to C_{R_1}; the function $f(z)$ may be taken as an arbitrary constant or other function analytic on and within such small circles. The reasoning previously given in connection with Corollary 2 is valid without change. We formulate

Corollary 4. *Under the hypothesis of Corollary 3, let $f(z)$ be analytic on C. Then the series of interpolation of the form*

$$a_0 + a_1(z - \beta_1) + a_2(z - \beta_1)(z - \beta_2) + \cdots$$

which is the formal expansion of $f(z)$ in the points β_ν, converges maximally to $f(z)$ on C. If $f(z)$ is not defined in a particular point β_ν, an arbitrary value may be assigned to it there.

Corollary 1 has obvious application to the situation of Theorem 1

The method of proof of Corollary 3 and Corollary 4 is obviously capable of much broader use than we have hitherto suggested.

As a general remark, we note that it is sufficient in the proof of Theorem 2 if (4) is assumed to hold not on *every* closed limited point set interior to K but on all curves C_R belonging to a set everywhere dense in K

Theorem 2 gives at least theoretically a necessary and sufficient condition for the analyticity of a given function $f(z)$ on C, namely the uniform convergence in some region containing C in its interior of the formal expansion of $f(z)$

§7.3. A necessary condition for uniform convergence

The new part of the following theorem can be expressed so as to be a converse of Theorem 2:

Theorem 3. *Let C be a closed limited point set whose complement K is connected and regular. Let $w = \phi(z)$ map K onto the region $|w| > 1$ so that the points at infinity correspond to each other, and let Δ be the capacity of C. Let the points (1) have no limit point exterior to C. Then a necessary and sufficient condition that*

) analytic and single-valued on C we have for the polynomial ... and by interpolation to $f(z)$ in the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$,

$$\lim_{n\to\infty} p_n(z) = f(z)\,, \qquad \textit{uniformly for } z \textit{ on } C\,,$$

is that (4) *be valid uniformly on any closed limited point set interior to K. If this condition is fulfilled, the sequence $p_n(z)$ converges maximally to every such $f(z)$ on C.*

Before proceeding with the proof of Theorem 3, it is convenient for us to introduce a lemma. The proof of the lemma is quite easy, but the formulated statement is nevertheless useful for reference:

LEMMA. *Let C be an arbitrary closed limited point set whose complement K (i.e., with respect to the extended plane) is connected and regular. Let $\chi(z)$ be analytic in K (except possibly for branch points) although not necessarily single-valued and let $|\chi(z)|$ be single-valued in K. At infinity let each branch of $\chi(z)$ have a simple pole, with $|\chi'(\infty)| = 1$. Let $w = \Delta\phi(z)$ map the region K onto $|w| > \Delta$ so that the points at infinity correspond to each other and so that*

$$\Delta\,|\,\phi'(\infty)\,| = 1\,.$$

Then either (a) *the modulus $|\chi(z)|$ approaches Δ whenever z interior to K approaches C or* (b) *for some methods of approach $|\chi(z)|$ has a limit greater than Δ and (if $\chi(z) \neq 0$ in K) for other methods of approach $|\chi(z)|$ has a limit less than Δ.*

The function

$$\frac{\chi(z)}{\Delta\phi(z)} \tag{13}$$

is analytic in K even at infinity, if suitably defined there; the value at infinity is of modulus unity. To be sure, the function (13) need not be single-valued and may have branch points in K; nevertheless its modulus is single-valued in K. Then (Principle of Maximum) either the modulus of (13) is unity everywhere in K and the function (13) is a constant of modulus unity or the modulus possesses no maximum interior to K. In the latter case, for some methods of approach to C the modulus of (13) approaches values greater than unity. If $\chi(z)$ is different from zero interior to K, the modulus of (13) can have no minimum interior to K, so if the modulus of (13) is not unity at all points of K, for some methods of approach to C the modulus of (13) approaches values less than unity.

Whenever z interior to K approaches C, the function $\Delta\,|\,\phi(z)\,|$ approaches the value Δ, so whenever z approaches C and the modulus of (13) approaches unity, a value greater than unity, or a value less than unity, then the modulus of $\chi(z)$ approaches Δ, a value greater than Δ, or a value less than Δ respectively. The lemma is completely proved.

We return to the proof of Theorem 3. The sufficiency of (4) is contained in Theorem 2, so we need consider only the necessity.

Let us first treat the case that K is simply connected. In the notation of (6), a suitably chosen branch of the function $\theta_n(z) \equiv [\omega_n(z)]^{1/(n+1)}$ is analytic except at infinity exterior to any C_ρ which contains in its interior all the $\beta_k^{(n)}$, and at infinity has a simple pole and the derivative unity. For sufficiently large n, each factor $z - \beta_k^{(n)}$ is uniformly limited in any closed finite region of the plane, and hence the functions $\theta_n(z)$ are similarly limited and form a normal family in the region exterior to C but not containing the point at infinity.

We are here using a slight extension of the usual notion of normal family, for points $\beta_k^{(n)}$ may lie in K and so the functions $\theta_n(z)$ are not necessarily analytic at each point of K. Nevertheless, in any given closed limited region belonging to K, at most a finite number of those functions fail to be analytic, so the entire set of functions has the characteristic properties of a normal family. In particular we remark that if (4) is valid at each point of K, then (4) is valid uniformly on any closed limited point set interior to K.

In the proof of Theorem 3, it happens that we need merely study the expansion of the function $f(z) = 1/(t - z)$, where t is suitably chosen exterior to C. For this particular function we have (§3.1)

$$f(z) - p_n(z) = \frac{\omega_n(z)}{\omega_n(t)(t - z)}.$$

If we introduce the notation

$$M_n = \max [\,|\,\omega_n(z)\,|\,,\ z \text{ on } C\,],$$

our hypothesis implies

$$\lim_{n\to\infty} M_n/[\omega_n(t)] = 0\,, \tag{14}$$

provided merely that t lies in K.

To prove Theorem 3 it is sufficient to show that every limit function in K of the family $\theta_n(z)$ coincides in K with $\Delta\phi(z)$, provided we choose $\Delta\phi'(\infty) = 1$, and by the properties of these functions at infinity it is therefore sufficient to show that every limit function in K of the family $\theta_n(z)$ has in K the modulus $\Delta\,|\,\phi(z)\,|$ We assume this not to be the case, and shall reach a contradiction.

Any limit function $\theta(z)$ of a subsequence $\theta_{n_k}(z)$ has a pole of the first order at infinity, and we have $\theta'(\infty) = 1$; indeed, the sequence $\{\theta_{n_k}(z) - z\}$ is analytic and converges uniformly throughout some neighborhood (of the extended plane) of the point at infinity. The function $\theta(z)$ cannot vanish in K, because at every point of K the $\theta_n(z)$ are uniformly bounded from zero for n sufficiently large. If $|\,\theta(z)\,|$ is not identically equal to $\Delta\,|\,\phi(z)\,|$ in K, it follows from the Lemma that $|\,\theta(z)\,|$ is actually *less* than Δ at some points interior to K. That is to say, we have for some sequence of indices n_k and for some point t in K,

$$\lim_{k\to\infty} |\,\theta_{n_k}(t)\,| = |\,\theta(t)\,| < \Delta\,.$$

There follow the inequalities

$$|\,\theta_{n_k}(t)\,| < \Delta\,, \qquad |\,\omega_{n_k}(t)\,| < \Delta^{n_k+1}\,, \tag{15}$$

for suitably large index. We choose this point t as the value in (14), and we apply the Lemma to the functions $\theta_{n_k}(z)$; we have

$$M_{n_k} \geqq \Delta^{n_k+1}. \tag{16}$$

Inequalities (15) and (16) are incompatible with (14), so the proof is complete in the case that K is simply connected.

In the case that K is multiply connected* the functions $\theta_n(z)$ need not be single-valued in K, and our use of normal families requires modification. There exist simply connected regions $K_1, K_2, \cdots$ of the extended plane interior to K such that every point of the extended plane in K lies interior to at least one region K_ν and any closed point set interior to K can be covered by a finite number of regions K_ν. The functions $\theta_n(z)$ still form a normal family interior to K in the sense that from any infinite sequence of $\theta_n(z)$ can be extracted (by the diagonal process) a subsequence which converges throughout K uniformly on any closed set interior to K, provided different branches of a given $\theta_n(z)$ are permitted in the different regions K_ν. With this understanding, the Lemma and all the other discussion of Theorem 3 are valid. The proof is complete.

Theorem 3 for the case that K is simply connected is due to Kalmár [1926], except that Kalmár replaces condition (4) by the condition

$$\lim_{n\to\infty} [(z - \beta_1^{(n)})(z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)})]^{1/(n+1)} = \Delta\,\phi(z)$$

uniformly on any closed limited point set interior to K. The fact that this latter condition is implied by (4) when K is simply connected is not difficult; compare the proof of Theorem 2, Corollary 2. Condition (4) is ordinarily simpler to apply than Kalmár's condition. The present proof differs materially from that of Kalmár; the proof can be considerably simplified [Walsh, 1933e] if we replace in the hypothesis the condition of uniform convergence of $p_n(z)$ to $f(z)$ on C by the condition of maximal convergence of $p_n(z)$ to $f(z)$ on C, whenever $f(z)$ is single-valued and analytic on C.

§7.4. Further conditions for maximal convergence

If the condition of Theorem 2 is fulfilled, it follows that the lemniscate $L_R^{(n)}$: $|\omega_n(z)| = (\Delta R)^{n+1}$ approaches uniformly the locus C_R: $|\phi(z)| = R$ for every $R > 1$. Indeed, let this locus C_R be enclosed in the region or regions bounded by C_{R_1} and C_{R_2}, $1 < R_1 < R < R_2$. On C_{R_1} we have by the uniformity of (4) the inequality $|\omega_n(z)| < (\Delta R)^{n+1}$ for n sufficiently large, and on C_{R_2} we have for n sufficiently large $|\omega_n(z)| > (\Delta R)^{n+1}$. That is to say (compare §4.1), every point of C_{R_1} is interior to $L_R^{(n)}$ and every point of C_{R_2} is exterior to $L_R^{(n)}$. The lemniscate $L_R^{(n)}$ lies in the region or regions between and bounded by C_{R_1} and C_{R_2}, and separates the loci C_{R_1} and C_{R_2}.

* The two cases can be readily handled together by the use of normal families of *harmonic* functions.

Reciprocally, if the points $\beta_k^{(n)}$ have no limit point in K and if the lemniscate $L_R^{(n)}$: $|\omega_n(z)| = (\Delta R)^{n+1}$ approaches uniformly the locus C_R for every R, the condition of Theorem 2 is fulfilled. If $R > 1$ is arbitrary, the lemniscate $L_{R_1}^{(n)}$, $1 < R_1 < R$, lies interior to C_R for n sufficiently large. Then on and exterior to C_R we have

$$\varliminf_{n\to\infty} |\omega_n(z)|^{1/(n+1)} \geqq \Delta R_1, \qquad \varliminf_{n\to\infty} |\omega_n(z)|^{1/(n+1)} \geqq \Delta R.$$

By the Lemma used in the proof of Theorem 3, where the present C_R plays the rôle of the previous C, it follows that every limit function (by Hurwitz's theorem necessarily non-vanishing in K) of the sequence $[\omega_n(z)]^{1/(n+1)}$ coincides with $\Delta\phi(z)$ exterior to C_R, in the extended sense described in §7.3, and this is true for every $R > 1$. The uniformity of approach in (4) follows at once.

It is not true that if the condition of Theorem 2 is fulfilled, then the lemniscate $|\omega_n(z)| = \Delta^{n+1}$ necessarily approaches the boundary of C uniformly, nor is the latter true even if C is bounded by an analytic Jordan curve. In fact, under the condition of Theorem 1, the lemniscate $|z^{n+1} - 1| = 1$ clearly passes through the origin, no matter what n may be. Compare Faber [1920].

As a complement to Theorems 2 and 3 we prove

THEOREM 4. *Let C, $\phi(z)$, and Δ satisfy the hypothesis of Theorem 2, and let the points (1) have no limit point exterior to C. If we introduce the notation of (6) and set*

$$M_n = \max [|\omega_n(z)|, z \text{ on } C],$$

then

$$\lim_{n\to\infty} M_n^{1/(n+1)} = \Delta \tag{17}$$

is a necessary and sufficient condition that (4) should hold uniformly on any closed limited point set interior to K, and hence is a necessary and sufficient condition for uniform or maximal convergence of the sequence of polynomials $p_n(z)$ of interpolation to an arbitrary function $f(z)$ single-valued and analytic on C.

If condition (17) is satisfied, the normal family of functions

$$[\omega_n(z)]^{1/(n+1)}/\phi(z) \tag{18}$$

which are all analytic (although not necessarily single-valued) in K even at infinity and have $M_n^{1/(n+1)}$ as an upper bound to their respective moduli in K, can have no limit function whose modulus in K is greater than Δ. But all the functions (18) have the value Δ at infinity, so every limit function of the set (18) is identically equal to Δ in K, which implies (4) as stated. If K is multiply connected, a convention like that of §7.3 must be introduced relative to convergence of a subset of functions (18).

Conversely, let (4) hold as stated; we are to establish (17). From the Lemma

we have $M_n \geqq \Delta^{n+1}$, so if (17) does not hold we have for suitable $\Delta_1 > \Delta$ and for a suitable infinite sequence of indices n_k

$$M_{n_k} \geqq \Delta_1^{n_k+1} > \Delta^{n_k+1} . \tag{19}$$

It follows from (4) that for suitably chosen t in K and arbitrary Δ_2, $\Delta < \Delta_2 < \Delta_1$, we have for sufficiently large index n,

$$| \omega_n(t) | \leqq \Delta_2^{n+1} < \Delta_1^{n+1} . \tag{20}$$

Equation (14) is a consequence of (4) by Theorem 2, and (19) and (20) are incompatible with (14). Theorem 4 is established.

Condition (17) was first introduced by Fekete in an unpublished paper, in a situation of somewhat less generality than here and by the use of other methods; the present theorem is due to Walsh [1934c] and to Fekete, in a paper also unpublished.

§7.5. Uniform distribution of points

As we may see by reference to §§4.3 and 7.2, interpolation in points $\beta_k^{(n)}$ which lie on the boundary of the given point set C is particularly interesting. In order to study that subject more thoroughly we consider the general topic of uniform distribution of points.

Let the interval $0 \leqq x \leqq 1$ be denoted by $\mathfrak{J}$ and let points

$$x_{n1}, x_{n2}, \cdots , x_{nn} \tag{21}$$

lie on $\mathfrak{J}$. Let $c \leqq x \leqq d$ be an arbitrary subinterval $\mathfrak{J}_1$ of $\mathfrak{J}$ and let $N_n(c, d)$ denote the number of points (21) which lie on $\mathfrak{J}_1$. The points (21) are said to be *uniformly distributed* on $\mathfrak{J}$ if and only if we have

$$\lim_{n\to\infty} \frac{N_n(c, d)}{n} = d - c , \tag{22}$$

no matter what subinterval $\mathfrak{J}_1$ may be chosen. The concept and the following properties are due to Weyl [1916; compare also Pólya and Szegö, 1925]. In particular we note that the points of equal subdivision $x_{nk} = k/n$ or $x_{nk} = (k - 1)/n$ are uniformly distributed on $\mathfrak{J}$, because $\mathfrak{J}_1$ is of length $d - c$, and we have

$$d - c - \frac{1}{n} < \frac{N_n(c, d)}{n} \leqq d - c + \frac{1}{n} ;$$

the points k/n or $(k - 1)/n$ are called *equidistributed* on $\mathfrak{J}$.

From the definition of uniform distribution follows: *a necessary and sufficient condition that the points* (21) *be uniformly distributed on* $\mathfrak{J}$ *is that for every function* $f(x)$ *piecewise continuous on* $\mathfrak{J}$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{nk}) = \int_0^1 f(x)\, dx . \tag{23}$$

The sum in the left-hand member of (23) is precisely a Riemann sum for the integral in the right-hand member if we have $x_{nk} = k/n$.

Let $\mathfrak{I}_1$ be arbitrary and let us choose

$$f(x) \equiv \begin{cases} 1, & c \leqq x \leqq d, \\ 0, & 0 \leqq x < c, \quad d < x \leqq 1. \end{cases} \tag{24}$$

Equations (22) and (23) are identical here, so (23) implies (22).

Conversely, let (22) hold. An arbitrary function $f(x)$ piecewise continuous on $\mathfrak{I}$ can be uniformly approximated on $\mathfrak{I}$ by a function $f_1(x)$ which is piecewise constant:

$$| f(x) - f_1(x) | < \epsilon, \qquad x \text{ on } \mathfrak{I}, \tag{25}$$

where $\epsilon > 0$ is preassigned, from which we may write

$$-\epsilon < \int_0^1 f_1(x)\, dx - \int_0^1 f(x)\, dx < \epsilon. \tag{26}$$

The function $f_1(x)$ is the sum of a finite number of functions each constant on a subinterval of $\mathfrak{I}$ (such a subinterval may be a single point) and zero on the complementary set. For each of these functions the equation analogous to (23) is valid, hence we may write

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_1(x_{nk}) = \int_0^1 f_1(x)\, dx.$$

When n is sufficiently large, we have

$$-\epsilon < \frac{1}{n} \sum_{k=1}^{n} f_1(x_{nk}) - \int_0^1 f_1(x)\, dx < \epsilon,$$

so from (25) and (26) we have

$$-3\epsilon < \frac{1}{n} \sum_{k=1}^{n} f(x_{nk}) - \int_0^1 f(x)\, dx < 3\epsilon,$$

which completes the proof of (23) and the theorem.

We have derived (22) as a consequence of (23) for all piecewise constant functions. It is similarly true that *the uniform distribution of the x_{nk} follows from* (23) *for all continuous functions*, as we now proceed to prove. We shall establish (23) for all piecewise continuous functions; it is sufficient to treat the function (24) in the case $c < d$. Let $\epsilon > 0$ be arbitrary and let us define a continuous function $F(x)$ which coincides with $f(x)$ of (24) for $c \leqq x \leqq d$, which is equal to zero in the points $x \leqq c - \epsilon$ or $x = 0$ according as $c - \epsilon > 0$ or $c - \epsilon \leqq 0$, and is linear between $x = c - \epsilon$ or $x = 0$ and $x = c$; which is equal to zero in the

points $x \geqq d + \epsilon$ or $x = 1$ according as $d + \epsilon < 1$ or $d + \epsilon \geqq 1$, and is linear between $x = d + \epsilon$ or $x = 1$ and $x = d$. We have

$$0 \leqq \int_0^1 F(x)\,dx - \int_0^1 f(x)\,dx \leqq \epsilon.$$

We have also

$$\frac{1}{n}\sum_{k=1}^{n} f(x_{nk}) \leqq \frac{1}{n}\sum_{k=1}^{n} F(x_{nk}).$$

But by hypothesis we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} F(x_{nk}) = \int_0^1 F(x)\,dx,$$

whence

$$\varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_{nk}) \leqq \int_0^1 f(x)\,dx + \epsilon. \tag{27}$$

We construct similarly a positive continuous function $F_1(x)$ which is never greater than $f(x)$, and obtain

$$\varliminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_{nk}) \geqq \int_0^1 f(x)\,dx - \epsilon,$$

which together with (27) implies (23) and hence (22).

We have already mentioned that the points $x_{nk} = k/n$ are uniformly distributed on $\mathfrak{F}$. Another illustration of importance is that of the numbers

$$x_{nk} = k\xi - [k\xi], \tag{28}$$

where the symbol $[q]$ denotes the largest integer not greater than q, and where ξ is an arbitrary irrational number. The numbers (28) are of particular interest because x_{nk} does not depend on n. We omit the proof that the numbers (28) are uniformly distributed on $\mathfrak{F}$. The theorem is due to Bohl, Sierpiński, and Weyl.

The definition of uniform distribution is to be altered in an obvious way if the fundamental interval $\mathfrak{F}$ is $a \leqq x \leqq b$ instead of $0 \leqq x \leqq 1$; more generally, the concept may refer to points on an arbitrary curve or on several curves C of the complex plane, where x is a parameter on C, $a \leqq x \leqq b$; equation (22) is to be replaced by

$$\lim_{n\to\infty} \frac{N_n(c, d)}{N_n(a, b)} = \frac{d - c}{b - a}, \qquad n = N_n(a, b).$$

The properties that we have derived extend without difficulty to this more general situation. Moreover, the definition is not changed if (a, b) and (c, d) are taken as open or semi-closed intervals.

§7.6. Interpolation in points uniformly distributed

Let us return now to the situation and notation of §§4.2–4.4. It is clear when C consists of a finite number of mutually exterior Jordan curves (pointed out by Fejér [1918] when C is a single Jordan curve) that the points $\zeta_m^{(n)}$ equidistributed on C used in §4.3, Theorem 3 can be replaced by arbitrary points uniformly distributed on C with respect to the parameter $u(\zeta)$. Equation (7) of §4.3 still persists, as does (see the method below) the equation $\lim_{n\to\infty} | \omega_n(z) |^{1/n} = e^g$, interior to C, uniformly on any closed set interior to C.

In particular there is a great advantage in using the points defined as in the present equation (28), for then the points $\zeta_m^{(n)} = \zeta_m$ are independent of n; the corresponding polynomial expansion of a given function found by interpolation in the points ζ_m is a series of interpolation (§3.2). For instance, if C is a single Jordan curve, we may set $\zeta_m = \psi(\epsilon^m)$, where ϵ is no root of unity but $| \epsilon | = 1$, and where $w = \phi(z)$, $z = \psi(w)$, is the usual mapping function. That is to say, $(2\pi i)^{-1} \log \epsilon$ is real and irrational, so by (28) the sequence $(2\pi i)^{-1} k \log \epsilon$, $k = 0, 1, 2, \cdots$, is uniformly distributed on the unit circle with respect to arc length as parameter; the points $\psi(\epsilon^m)$ are uniformly distributed on the boundary of C with respect to the parameter $u(\zeta)$. The introduction of these points $\zeta_m = \psi(\epsilon^m)$ is likewise due to Fejér, who thus obtained the first set of polynomials belonging to (§6.5) an arbitrary Jordan region; the polynomials are the set $1, (z - \zeta_1), (z - \zeta_1)(z - \zeta_2), \cdots$.

This application to the situation of §4.3, Theorem 3 of the concept of uniform distribution of points is obviously valid whether the Jordan curves composing C are analytic or not. If these curves are not analytic, still a new method for the study of this uniform distribution is of interest.

Let $G(x, y)$ be Green's function for K with pole at infinity, so that $G(x, y) - \log R$ is Green's function for the exterior of C_R with pole at infinity. If we replace C by C_R, the function $V(x, y)$ of §4.2 is replaced by $V_R(x, y) \equiv V(x, y)$. When R is sufficiently near unity, the locus C_R consists of a finite number of mutually exterior analytic Jordan curves, and we have

$$(29) \qquad V_R(x, y) = \int_0^{u_0} \log r \, du , \qquad du = \frac{1}{2\pi} \frac{\partial V}{\partial n} ds ,$$

for $z = x + iy$ exterior to C_R, the integral being taken over C_R. The differential du is, except for the factor $(2\pi)^{-1}$, the differential on C_R of a function $u(x, y)$ conjugate to $V_R(x, y)$ or to $V(x, y)$. This conjugate function is defined, locally single-valued, and harmonic at every finite point of K, and is continuous even on C if suitably defined there, as may be seen (§4.3) by mapping K onto a region bounded by analytic Jordan curves. When R approaches unity, the function $\log r = \log | z - \zeta |$ (for z fixed interior to K and ζ on C_R but ζ moving on a level curve $u =$ constant) approaches uniformly in ζ the corresponding function $\log | z - \zeta |$ with ζ on C. Even though C is non-rectifiable, the differential du has a meaning on C, and $\log r$ is a continuous function of u. The multiple-

valuedness of u presents no difficulty, for the curves u = constant are independent of the branch of u chosen, and only the differential du appears in the integral. That is to say, the integral in (29) has a meaning even when that integral is taken over C, with $V_R(x, y) \equiv V(x, y) \equiv G(x, y) + g$, provided u is defined on C by means of continuity; by the uniformity of the convergence of $\log r$ when R approaches unity, equation (29) is itself valid for z in K. The validity of (29) in this form is sufficient for our application to the situation of §§4.2 and 4.3 of the concept of uniform distribution. Equation (7) of §4.3 is thus valid whenever the points $\zeta_m^{(n)}$ are uniformly distributed on C with respect to the parameter u.

Even if C does not consist of a finite number of Jordan curves, but if K (the infinite region bounded by the closed limited set C) is of finite connectivity and regular, equation (29) is valid if the integral is taken over C_R. For certain suitably chosen curves u = constant, the function $u(x, y)$ can be defined by continuity in points of C, the integral in (29) can be defined on C, the corresponding equation (29) is valid for z in K, and the integral can still be evaluated by Riemann sums [compare Jessen, 1934]. Hence points $\zeta_m^{(n)}$ exist on C with the fundamental property of §4.3, equation (7). But it may be difficult to determine these points $\zeta_m^{(n)}$ effectively, so we shall not elaborate this remark.

We now study necessary conditions concerning interpolation in points which lie on the boundary of a given point set. We shall prove

THEOREM 5. *Let C be a closed limited point set of the finite z-plane consisting of a finite number of mutually exterior Jordan regions. Let the points $\beta_k^{(n)}$ lie on the boundary B of C; a necessary and sufficient condition that the sequence of polynomials $p_n(z)$ of degree n found by interpolation to an arbitrary function $f(z)$ analytic on C in the points $\beta_k^{(n)}$ converges uniformly to $f(z)$ on C, is that the points $\beta_k^{(n)}$ be uniformly distributed on C with respect to the parameter u (§4.3). If this condition is satisfied, the sequence $p_n(z)$ converges maximally to $f(z)$ on C.*

The sufficiency of this condition follows from Theorem 2 and the discussion given above, and is due to Fejér if C is a single Jordan region; the necessity is now to be proved by methods introduced by Kalmár [1926] in his proof of the theorem for the case that C consists of a single Jordan region.

It follows from § 7.3, Theorem 3 that the condition

$$\lim_{n\to\infty} | \omega_n(z) |^{1/(n+1)} = \Delta | \phi(z) | ,$$

uniformly on any closed limited set interior to the complement K of C, in the notation of (4) and (6), is fulfilled. We write this condition in the form

$$(30) \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{n+1} \log r_{nk} = V(x, y) = \int_0^{u_0} \log r \, du ,$$

$$du = \frac{1}{2\pi} \frac{\partial V}{\partial n} ds , \qquad u_0 = 1 ,$$

where $V(x, y) - \log \Delta$ is Green's function for K with pole at infinity, and where we have $r_{nk} = |z - \beta_k^{(n)}|$, $r = |z - \beta|$, $z = x + iy$, the point β being on B and corresponding to the variable u. If C is not bounded by *analytic* Jordan curves, this integral over B is to be taken as obtained by a limiting process in the manner already described from (29).

Equation (30) with the second member omitted expresses the fact that the limit of the average of the quantities $\log r_{nk}$ equals the integral of $\log r$, where r is measured from an arbitrary point z of K; that is, the limit of the average of $\log r$ in the points $\beta_k^{(n)}$ equals the integral of $\log r$. Our conclusion follows (§7.5) provided the corresponding equation is true not merely for the function $\log r$ but for every function $f(\beta)$ continuous on B. Such an equation as

$$\lim_{n\to\infty} \frac{1}{n+1} f_j(\beta_k^{(n)}) = \int_0^{u_0} f_j(\beta)\, du$$

holding for continuous functions $f_j(\beta)$ holds for any linear combination of them, and also for any function which can be uniformly approximated by such a linear combination. It is then sufficient to prove (as we proceed to do) that any function $f(\beta)$ continuous on B can be uniformly approximated on B by a linear combination of functions $f_j(\beta) \equiv \log|z_j - \beta|$, where z_j lies in K.

The following lemmas are valid in the general case under Theorem 5, whether the Jordan curves composing B are analytic or not.

LEMMA I. *Let K be an infinite region whose boundary B consists of a finite number of mutually exterior finite Jordan curves. Let C be the complement of K. Let $U(x, y)$ be an arbitrary function continuous on B. Let $\epsilon > 0$ be arbitrary. Then there exists a harmonic polynomial $P(x, y)$ such that we have*

$$(31) \qquad |U(x, y) - P(x, y)| < \epsilon, \qquad (x, y) \text{ on } B.$$

A function harmonic in a closed limited Jordan region can be uniformly approximated there by a harmonic polynomial; this follows from §1.6, Theorem 8, using the method of approximating by a polynomial in z an analytic function whose real part is the given harmonic function, and then taking real parts respectively of the analytic function and of the polynomial in z. It is also sufficient, by the method of proof of §2.4 Theorem 5, if the given function is harmonic interior to the Jordan region, continuous in the corresponding closed region.

There exists a function $U(x, y)$ harmonic interior to C, continuous on the closed set C, coinciding with the given function $U(x, y)$ on B. This function $U(x, y)$ can be uniformly approximated by a harmonic polynomial in *each* of the closed Jordan regions comprising C, and it follows by the method of §2 8 that $U(x, y)$ can be *simultaneously* uniformly approximated by a single harmonic polynomial in all of these closed Jordan regions. This implies (31).

LEMMA II. *Let K be an infinite region whose boundary B consists of a finite number of mutually exterior finite Jordan curves. Let C be the complement of K.*

Let $U(x, y)$ be an arbitrary function continuous on B. Let $\epsilon > 0$ be arbitrary. Then there exists a finite set of points $\zeta_0, \zeta_1, \cdots, \zeta_n$ interior to K such that we have

$$\left| U(x,y) - \sum_{k=0}^{n} A_k \log |z - \zeta_k| \right| < \epsilon, \qquad z \text{ on } B, \tag{32}$$

where the A_k are constants.

In the proof of Lemma II we may, by Lemma I, assume $U(x, y)$ to be harmonic on the closed set C. Let us set $F(z) \equiv e^{U+iU_1}$, where U_1 is conjugate to U on C; then $F(z)$ is analytic and different from zero on C. If $\delta > 0$ is arbitrary, there exists a polynomial $p(z)$ in z such that we have on C

$$\left| \frac{p(z) - F(z)}{F(z)} \right| < \delta, \qquad \left| \left| \frac{p(z)}{F(z)} \right| - 1 \right| < \delta.$$

Thus there exists a polynomial $p(z)$ with the property

$$e^{-\epsilon} < \left| \frac{p(z)}{F(z)} \right| < e^{\epsilon}, \qquad z \text{ on } C, \tag{33}$$

and this inequality can be expressed in the form

$$| \log | F(z) | - \log | p(z) || < \epsilon, \qquad z \text{ on } C. \tag{34}$$

The polynomial $p(z)$ may be written

$$p(z) \equiv A(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n),$$

and it follows from (33) that all the points ζ_k lie exterior to C.

The constant $\log | A |$ can be uniformly approximated on C by a function of the form $A_0 \log | z - \zeta_0 |$, for we have uniformly for z on C

$$\lim_{|\zeta| \to \infty} \frac{\log | A | \log | z - \zeta |}{\log | \zeta |} = \log | A |;$$

it is sufficient to choose $| \zeta_0 |$ sufficiently large, with $A_0 = \log | A | / \log | \zeta_0 |$. Inequality (34) asserts that on C the function $U(x, y) \equiv \log | F(z) |$ can be uniformly approximated by $\log | p(z) |$; but $\log | p(z) |$ can be uniformly approximated on C by a sum of the form that appears in (32), so Lemma II follows at once.

Lemma II with a slight change of notation yields now the necessity of the condition of Theorem 5.

§7.7. Points of interpolation with extremal properties

THEOREM 6. *Let C be a closed limited point set whose complement K is connected and regular. Let $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$ be a set of $n + 1$ points $z_k^{(n)}$ of C such that the modulus of the product (Vandermonde determinant)*

$$V_n(z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}) \equiv \prod_{i<j=1}^{j=n+1} (z_i^{(n)} - z_j^{(n)})$$

is greatest. If $f(z)$ is single-valued and analytic on C, then the sequence of polynomials $p_n(z)$ of respective degrees n defined by interpolation to $f(z)$ in the points $\beta_k^{(n)}$ converges maximally to $f(z)$ on C. Consequently equations (4) *and* (17) *are valid.*

The function $|V_n(z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)})|$ is continuous when the $z_k^{(n)}$ lie on the closed limited point set C, and so possesses a maximum for the $z_k^{(n)}$ on C. The points $z_k^{(n)} = \beta_k^{(n)}$ which furnish this maximum need not be unique, but *any* determination suffices for the present application; the points $\beta_k^{(n)}$ are obviously all distinct.

Let us take Lagrange's interpolation formula for interpolation to the function $F(z)$ in the form of §3.1, equation (3):

$$P_n(z) = \sum_{k=1}^{n+1} F(\beta_k^{(n)}) \frac{\omega_n(z)}{(z - \beta_k^{(n)})\, \omega_n'(\beta_k^{(n)})}, \tag{35}$$

where $\omega_n(z)$ is defined by (6). From the definition of the points $\beta_k^{(n)}$ follows the inequality

$$\left| \frac{\omega_n(z)}{(z - \beta_k^{(n)})\, \omega_n'(\beta_k^{(n)})} \right| \leqq 1, \qquad z \text{ on } C, \tag{36}$$

because the function $(z - \beta_1^{(n)})(z - \beta_2^{(n)}) \cdots (z - \beta_{k-1}^{(n)})(z - \beta_{k+1}^{(n)}) \cdots (z - \beta_{n+1}^{(n)})$ is a factor which appears (for $z = \beta_k^{(n)}$) in the formula for V_n; the value $\beta_k^{(n)}$ appears in V_n only in this factor; this factor considered as a function of z takes on its maximum absolute value on C for $z = \beta_k^{(n)}$; the last statement is simply (36).

Let $R < \rho$ be given, where ρ has the usual significance (§4.7); there exist polynomials $\pi_n(z)$ of respective degrees n such that we have

$$|f(z) - \pi_n(z)| \leqq M/R^n, \qquad z \text{ on } C. \tag{37}$$

The polynomial $p_n(z)$ of degree n of interpolation to $f(z)$ in the points $\beta_k^{(n)}$ is the polynomial $P_n(z)$ of degree n of interpolation to the function $f(z) - \pi_n(z)$ in the points $\beta_k^{(n)}$ plus the polynomial of degree n of interpolation to the function $\pi_n(z)$ in the points $\beta_k^{(n)}$; the last polynomial is unique and hence coincides with $\pi_n(z)$ itself Then we have by (35), (36), and (37),

$$|f(z) - p_n(z)| \leqq |f(z) - \pi_n(z)| + |P_n(z)| \leqq M/R^n + (n+1)M/R^n, \qquad z \text{ on } C.$$

Thus we have

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - p_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R,$$

the maximal convergence of the sequence $p_n(z)$ to $f(z)$ on C; the remainder of the theorem [proved in another way by Fekete] follows from Theorems 3 and 4.

Theorem 6 is due, in a somewhat less general form, to Fekete [1926]. In particular if C is the region $|z| \leqq 1$, then the points $\beta_k^{(n)}$ may be chosen as the

$(n+1)$-st roots of unity considered by Runge; for this case Theorem 6 is included in Theorem 1.

It is clear that the conclusion of Theorem 6 persists if the $z_k^{(n)} = \beta_k^{(n)}$ are not defined so as to give to V_n its greatest value, but for instance are chosen on C for each n so that the expression

$$(38)\quad \max \sum_{k=1}^{n+1} \left| \frac{(z - z_1^{(n)}) \cdots (z - z_{k-1}^{(n)})(z - z_{k+1}^{(n)}) \cdots (z - z_{n+1}^{(n)})}{(z_k^{(n)} - z_1^{(n)}) \cdots (z_k^{(n)} - z_{k-1}^{(n)})(z_k^{(n)} - z_{k+1}^{(n)}) \cdots (z_k^{(n)} - z_{n+1}^{(n)})} \right|, \quad z \text{ on } C,$$

is least; these points $\beta_k^{(n)}$ are somewhat similar to points introduced in a different connection by Leja [1934].

It follows in this case from (38) and from our discussion of (36) that the expression (38) when the $z_k^{(n)}$ are replaced by the $\beta_k^{(n)}$ is not greater than $n + 1$; the proof of maximal convergence goes through as before. Various other point sets with minimal properties will occur to the reader; thus the $\beta_k^{(n)}$ can be chosen on C for each n such that the maximum on C of the sum of the p-th powers $(p > 0)$ of the $n + 1$ absolute values which appear in (38) is least, or such that the maximum on C of a suitable linear combination of these p-th powers is least. The conclusion of Theorem 6 holds even in this case.*

If in Theorem 6 the restriction that K be regular is omitted, but if C possesses infinitely many points, it is still true (by §4.9) that the sequence $p_n(z)$ converges uniformly to $f(z)$ on C, and the corresponding remark holds for the other point sets $\beta_k^{(n)}$ that we have mentioned. Indeed, the sequence $p_n(z)$ still converges maximally (§4.9) to $f(z)$ on C.

The following is an application of Theorem 4:

Let the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$ be chosen on C (satisfying the hypothesis of Theorem 2) such that

$$M_n = \max |(z - \beta_1^{(n)})(z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)})|, \qquad z \text{ on } C,$$

is not greater than the corresponding expression for any other set of $n + 1$ points on

* As another example, let the points $z_k^{(n)} = \beta_k^{(n)}$ be chosen on C so that the greatest of the $n + 1$ numbers

$$L_k^{(n)} = \max \left| \frac{(z - z_1^{(n)}) \cdots (z - z_{k-1}^{(n)})(z - z_{k+1}^{(n)}) \cdots (z - z_{n+1}^{(n)})}{(z_k^{(n)} - z_1^{(n)}) \cdots (z_k^{(n)} - z_{k-1}^{(n)})(z_k^{(n)} - z_{k+1}^{(n)}) \cdots (z_k^{(n)} - z_{n+1}^{(n)})} \right|, \quad z \text{ on } C,$$

is least. The conclusion of Theorem 6 is still true. However, it follows from our discussion of (36) that for $z_k^{(n)} = \beta_k^{(n)}$ every $L_k^{(n)}$ is less than or equal to unity, and it is obvious by the definition of the $L_k^{(n)}$ that every $L_k^{(n)}$ is greater than or equal to unity. Thus it is clear that the points $\beta_k^{(n)}$ of Theorem 6 satisfy the present extremal requirements, and it is not clear that other points can satisfy those requirements.

An obvious sufficient condition for maximal convergence of the sequence of polynomials found by interpolation in the points $z_k^{(n)}$ to an arbitrary function analytic on C, whether the $z_k^{(n)}$ satisfy extremal properties or not, is

$$\lim_{n\to\infty} [1 + L_1^{(n)} + L_2^{(n)} + \cdots + L_{n+1}^{(n)}]^{1/n} = 1.$$

C. Then condition (17) *is satisfied.*

There exists some set of points $\gamma_1^{(n)}, \gamma_2^{(n)}, \cdots, \gamma_{n+1}^{(n)}$ on C such that

$$\lim_{n\to\infty} [\max |(z - \gamma_1^{(n)})(z - \gamma_2^{(n)}) \cdots (z - \gamma_{n+1}^{(n)})|, z \text{ on } C]^{1/(n+1)} = \Delta;$$

indeed Fekete's points (Theorem 6) satisfy this condition, by Theorem 4. Thus we have, by the present definition of M_n,

$$\overline{\lim_{n\to\infty}}\, M_n^{1/(n+1)} \leqq \Delta.$$

But by the Lemma of §7.3 we have $M_n^{1/(n+1)} \geqq \Delta$, so (17) follows.

Polynomials related to the one whose maximum modulus is M_n have been studied by Tchebycheff, Faber, Fekete, and others.

§7.8. Existence of polynomials converging maximally (Shen)

Shen [1935] has constructed a simple proof of the existence of a sequence of polynomials converging maximally to a given function $f(z)$ analytic on a closed limited point set C whose complement K is connected and regular. We shall present this proof, a proof of §4.5 Theorem 5 where the polynomials to be exhibited do not depend on R, which is entirely direct and relatively elementary although perhaps more artificial than the proof given in §4.5.

Let $f(z)$ be single-valued and analytic throughout the closed interior of C_R, and let R_1 be arbitrary, $1 < R_1 < R$. We use the notation of §7.7; in particular the points $z_k^{(n)} = \beta_k^{(n)}$ shall be chosen as the *Fekete points*, that is the points $z_k^{(n)}$ of C such that the modulus of the Vandermonde determinant

$$V_n(z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}) \equiv \prod_{i<j=1}^{j=n+1} (z_i^{(n)} - z_j^{(n)})$$

is greatest.

Let t be a parameter on C_{R_1}. The polynomial in z of degree n for interpolation to the function $1/(t - z)$ in the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$ is

$$p_n(z, t) = \sum_{i=1}^{n+1} \frac{1}{t - \beta_i^{(n)}} \frac{V(\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{i-1}^{(n)}, z, \beta_{i+1}^{(n)}, \cdots, \beta_{n+1}^{(n)})}{V(\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)})}; \tag{39}$$

we verify at once that the right-hand member represents a polynomial of degree n in z, which at the point $z = \beta_i^{(n)}$ takes on the prescribed value $1/(t - \beta_i^{(n)})$.

By the definition of the points $\beta_k^{(n)}$ we have for z on C (as in (36))

$$|V(\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{i-1}^{(n)}, z, \beta_{i+1}^{(n)}, \cdots, \beta_{n+1}^{(n)})| \leqq |V(\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)})|.$$

Consequently, if d_1 denotes the minimum distance from C to C_{R_1} we have by (39)

$$|p_n(z, t)| \leqq (n + 1)/d_1, \quad z \text{ on } C, \quad t \text{ on } C_{R_1}. \tag{40}$$

We set as usual $\omega_n(z) = (z - \beta_1^{(n)})(z - \beta_2^{(n)}) \cdots (z - \beta_{n+1}^{(n)})$, and the identity

$$\frac{1}{t - z} - p_n(z, t) = \frac{1}{t - z} \frac{\omega_n(z)}{\omega_n(t)}$$

is familiar (§3.1). From inequality (40) we have

$$\left| \frac{\omega_n(z)}{(t - z)\, \omega_n(t)} \right| \leqq (n + 2)/d_1\,, \qquad z \text{ on } C\,,\ t \text{ on } C_{R_1}\,,$$

and if d_2 denotes the maximum distance from a point of C to a point of C_{R_1},

(41) $$|\, \omega_n(z)/\omega_n(t) \,| \leqq (n + 2)d_2/d_1\,, \qquad z \text{ on } C\,,\ t \text{ on } C_{R_1}\,.$$

For t on C_{R_1} we have $|\, \phi(t) \,| = R_1$ by definition. The function of t which appears in the left-hand member of (41), when z is fixed, is analytic exterior to C_{R_1} and either vanishes identically or has a zero of order $n + 1$ at the point $t = \infty$. The function

$$\omega_n(z)\, [\phi(t)]^{n+1}/\omega_n(t)$$

is a function of t analytic except perhaps for branch points for t on or exterior to C_{R_1}, even at $t = \infty$, when z is fixed on C, and its modulus although not the function itself is necessarily single-valued. The following inequality holds for z fixed on C and for all t on C_{R_1}, hence holds for z fixed on C and for all t on or exterior to C_{R_1},

$$\left| \frac{\omega_n(z)\, [\phi(t)]^{n+1}}{\omega_n(t)} \right| \leqq (n + 2)\, \frac{d_2}{d_1}\, R_1^{n+1}\,.$$

In particular for t on C_R, $R > R_1$, we have $|\, \phi(t) \,| = R$,

(42) $$|\, \omega_n(z)/\omega_n(t) \,| \leqq (n + 2)\, \frac{d_2}{d_1} \left(\frac{R_1}{R}\right)^{n+1}, \qquad z \text{ on } C\,.$$

If $p_n(z)$ denotes the polynomial of degree n of interpolation to $f(z)$ in the points $\beta_k^{(n)}$, we have by (42)

$$f(z) - p_n(z) = \frac{1}{2\pi i} \int_{C_R} \frac{\omega_n(z)\, f(t)\, dt}{\omega_n(t)\, (t - z)}\,, \qquad z \text{ on } C\,,$$

$$\overline{\lim_{n\to\infty}}\, [\max\, |\, f(z) - p_n(z) \,|\,,\, z \text{ on } C]^{1/n} \leqq R_1/R\,.$$

By the arbitrariness of R_1 we may write

$$\overline{\lim_{n\to\infty}}\, [\max\, |\, f(z) - p_n(z) \,|\,,\, z \text{ on } C]^{1/n} \leqq 1/R\,,$$

and this is essentially the same as the statement of §4.5, Theorem 5.

It is to be noticed that this proof employs only the most elementary properties of the points $\beta_k^{(n)}$, and of the loci C_R.

This proof does not apply to the general case that K is irregular, for it may be that all loci C_{R_1} for R_1 sufficiently near unity have points in common with C itself; the distance d_1 is zero.

§7.9. A synthesis of interpolation and Tchebycheff approximation

It is obvious from the various results already proved on interpolation and Tchebycheff approximation that the relation between these two theories is very close.

Let $f(z)$ be analytic on a closed limited point set C, and let $C^{(n)}$ be a closed subset of C with at least $n + 1$ points. Let $p_n(z)$ be the polynomial of degree n of best approximation to $f(z)$ on $C^{(n)}$ in the sense of Tchebycheff. Then $p_n(z)$ exists and is unique, and the sequence $p_n(z)$ may converge to $f(z)$ on C. If $C^{(n)}$ contains precisely $n + 1$ points, then $p_n(z)$ coincides with $f(z)$ in those points and is found by interpolation. If $C^{(n)}$ coincides with C, the polynomial $p_n(z)$ is the Tchebycheff polynomial previously studied (§5.1). If in the usual notation $C^{(n)}$ is a point set Γ independent of n and if C coincides with Γ_R, then the convergence of the sequence $p_n(z)$ to $f(z)$ on C is implied by §5.1, Theorem 2.

We shall now indicate a few situations, different from those just mentioned, where non-trivial point sets $C^{(n)}$ with some degree of arbitrariness lead to sequences $p_n(z)$ converging to $f(z)$.

Theorem 7. *Let C be the region $|z| \leqq 1$, let $f(z)$ be analytic on C, and let the point set $C^{(n)}$ contain a set of $n + N_n$ points z, with $z^{n+N_n} = A_n^{n+N_n}$, $N_n \geqq 1$, $|A_n| \leqq 1$, but let $C^{(n)}$ be contained in the set $|z| \leqq |A_n|$. Then the sequence of polynomials $p_n(z)$ of respective degrees n of best approximation to $f(z)$ on $C^{(n)}$ in the sense of Tchebycheff converges maximally to $f(z)$ on C*

In the proof of Theorem 7 we shall find it convenient to have for reference the following

Lemma. *If $q(z)$ is a polynomial of degree n such that in each of the points $z^{N+1} = A^{N+1}$, $N \geqq n$, we have $|q(z)| \leqq \Delta_1$, then for $|z| = |A|$ we have*

$$|q(z)| \leqq (n + 1)\Delta_1. \tag{43}$$

The proof is by Lagrange's interpolation formula. Let ω be a primitive $(N + 1)$-st root of unity, and let us set $q(A\omega^k) = q_k$. Then we have

$$\begin{aligned} q(z) &= \sum_{k=1}^{N+1} \frac{z^{N+1} - A^{N+1}}{z - A\omega^k} \, \frac{q_k}{(N+1)A^N\omega^{kN}} \\ &= \sum_{k=1}^{N+1} \frac{q_k}{(N+1)A^N\omega^{kN}} \, [z^N + A\omega^k z^{N-1} + A^2\omega^{2k}z^{N-2} + \cdots + A^N\omega^{kN}]. \end{aligned}$$

Since $q(z)$ is of degree n, we can neglect powers of z higher than the n-th:

$$q(z) = \sum_{k=1}^{N+1} \frac{q_k}{(N+1) A^N \omega^{kN}} [A^{N-n}\omega^{k(N-n)}z^n + A^{N-n+1}\omega^{k(N-n+1)}z^{n-1} + \cdots + A^N\omega^{kN}]$$

$$= \sum_{k=1}^{N+1} \frac{q_k}{(N+1) A^n \omega^{kn}} [z^n + A\omega^k z^{n-1} + \cdots + A^n \omega^{kn}].$$

For $| z | = | A |$ we now verify (43).*

Let $f(z)$ be analytic for $| z | < \rho > 1$ but have a singularity on the circle $| z | = \rho$. Let $R < \rho$ be arbitrary, $R > 1$. If $P_n(z)$ is the sum of the first $n + 1$ terms of the Taylor development of $f(z)$ about the origin, we have for $| z | = r \leqq 1$

$$| f(z) - P_n(z) | \leqq Mr^n/R^n, \tag{44}$$

where M is independent of r and of n. Then for $| z | = | A_n |$ we have

$$| f(z) - P_n(z) | \leqq M | A_n |^n/R^n,$$

whence by the definition of $p_n(z)$ we have for z on $C^{(n)}$

$$| f(z) - p_n(z) | \leqq M | A_n |^n/R^n, \qquad | P_n(z) - p_n(z) | \leqq 2M | A_n |^n/R^n.$$

The Lemma now implies the inequality

$$| P_n(z) - p_n(z) | \leqq 2(n+1)M | A_n |^n/R^n, \qquad | z | = | A_n |,$$

whence by the Lemma of §4.6,

$$| P_n(z) - p_n(z) | \leqq 2(n+1)M/R^n, \qquad z \text{ on } C. \tag{45}$$

Inequalities (44) and (45) imply that the sequence $p_n(z)$ converges maximally to $f(z)$ on C:

$$\overline{\lim_{n\to\infty}} [\max | f(z) - p_n(z) |, z \text{ on } C]^{1/n} \leqq 1/R.$$

Let us denote the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta_{n+1}^{(n)}$ of Theorem 6 by the term *Fekete points of C of order n*. Then we shall prove:

THEOREM 8. *Let C be a closed limited point set whose complement K is connected and regular, and let $C^{(n)}$ be a closed set which belongs to C and which contains the Fekete points of C of some order $N_n \geqq n$. Then the sequence of polynomials $p_n(z)$ of respective degrees n of best approximation to $f(z)$ (assumed analytic on C) on $C^{(n)}$ in the sense of Tchebycheff converges maximally to $f(z)$ on C.*

Here too it is convenient first to prove the corresponding

* We note in passing that the following can be proved as in the treatment of (37):
If $f(z)$ is defined on C': $| z | = | A |$, and if there exists a polynomial $P(z)$ of degree n such that we have $| f(z) - P(z) | \leqq \epsilon_n$ on C', then the polynomial $p(z)$ of degree n that coincides with $f(z)$ in the points $z^{n+1} = A^{n+1}$ satisfies the inequality $| f(z) - p(z) | \leqq (n+2)\epsilon_n$ on C'.

LEMMA. *Let Γ be a closed limited point set, let $q(z)$ be a polynomial of degree $n > 0$ which satisfies the inequality $| q(z) | \leqq \Delta_1$ on a set of Fekete points of Γ of order $N \geqq n$. Then on Γ we have*

$$| q(z) | \leqq 2n\Delta_1 .$$

Let us set $N = kn + p$, $n > p \geqq 0$, where k and p are integers. In the Fekete points we have the inequalities $| q(z) | \leqq \Delta_1$, $| q(z) |^k \leqq \Delta_1^k$. The polynomial $[q(z)]^k$ is of degree N, so (by (35) and (36) with the reasoning used in the proof of Theorem 6) we have for z on Γ

$$| q(z) |^k \leqq (N + 1)\Delta_1^k .$$

We can therefore write for z on Γ

$$| q(z) | \leqq (N + 1)^{1/k}\Delta_1 \leqq [(k + 1)n]^{1/k}\Delta_1 \leqq 2n\Delta_1 ,$$

because the inequality $(k + 1)^{1/k} \leqq 2$ holds for all positive integers k.

Theorem 8 can now be proved by the method previously used. There exist polynomials $P_n(z)$ of respective degrees n such that we have for z on C, hence for z on $C^{(n)}$,

$$| f(z) - P_n(z) | \leqq M/R^n , \tag{46}$$

where $R < \rho$ (usual notation) is arbitrary. Thus we have for z on $C^{(n)}$,

$$| f(z) - P_n(z) | \leqq M/R^n , \qquad | P_n(z) - p_n(z) | \leqq 2M/R^n .$$

From the Lemma follows the inequality

$$| P_n(z) - p_n(z) | \leqq 4nM/R^n ,$$

and this together with (46) yields the theorem.

Theorem 8 holds also if K is not regular, and the Fekete points may be replaced by the other sets of points mentioned in §7.7. Theorem 8 also holds if C is a point set bounded by a curve $\Gamma_{R'}$, and if the point set $C^{(n)}$ belongs to some Γ_{R_n}, $1 < R_n \leqq R_0 < R'$, provided the set $C^{(n)}$ contains the Fekete points of Γ_{R_n} of some order $N_n \geqq n$. This last generalization of Theorem 8 essentially contains Theorem 7 as a limiting case. The reader can easily prove these generalizations by the methods already used. These generalizations are all contained in the following theorem, provided we omit (as is allowable) in Theorem 9 the requirement that the complement of C be regular.

THEOREM 9. *Let C be a closed limited point set whose complement is connected and regular. Let the closed point set $C^{(n)}$ belong to C or to the closed interior of C_{R_n}, $1 < R_n \leqq \tau$. Let the inequality $| q_n(z) | \leqq 1$, z on $C^{(n)}$, where $q_n(z)$ is an arbitrary polynomial of degree n, imply $| q_n(z) | \leqq Q_n^{(\sigma)}(z)$ for z on every C_σ, $\sigma > \tau$, and where for every such σ*

$$\lim_{n\to\infty} [R_n^n Q_n^{(\sigma)}(z)]^{1/n} = \sigma , \qquad \textit{uniformly for } z \textit{ on } C_\sigma . \tag{47}$$

If $f(z)$ is analytic on and within C_τ, then the sequence of polynomials $p_n(z)$ of degree n of best approximation to $f(z)$ on $C^{(n)}$ in the sense of Tchebycheff converges maximally to $f(z)$ on C_τ.

The condition of Theorem 9 is satisfied whenever $C^{(n)}$ is the set C_{R_n}, for under such circumstances the inequality $| q_n(z) | \leqq 1$, z on $C^{(n)}$, implies $| q_n(z) | \leqq \sigma^n/R_n^n$ for z on C_σ. Consequently the condition of Theorem 9 is satisfied also whenever $C^{(n)}$ is a Fekete set of order $N_n \geqq n$ on C_{R_n}. Let us proceed to the proof of the theorem.

Let $f(z)$ be single-valued and analytic for z interior to C_ρ, but not for z interior to any $C_{\rho'}$, $\rho' > \rho$. Let $R < \rho$ be arbitrary, $R > \tau$. There exist polynomials $P_n(z)$ of respective degrees n such that we have

$$| f(z) - P_n(z) | \leqq M_1/R^n, \qquad z \text{ on } C.$$

For z on C_{R_n} we have therefore (method of §4.7)

$$| f(z) - P_n(z) | \leqq MR_n^n/R^n,$$

where M depends on R and τ but not on R_n or n. Further inequalities follow immediately for z on $C^{(n)}$ (if $C^{(n)}$ belongs to C we set $R_n = 1$):

$$| f(z) - p_n(z) | \leqq MR_n^n/R^n, \qquad | P_n(z) - p_n(z) | \leqq 2MR_n^n/R^n.$$

By the hypothesis on $C^{(n)}$ we can now write

$$| P_n(z) - p_n(z) | \leqq 2MR_n^n Q_n^{(\sigma)}(z)/R^n, \qquad z \text{ on } C_\sigma.$$

The sequence $P_n(z)$ converges maximally to $f(z)$ on C_σ:

$$| f(z) - P_n(z) | \leqq M'\sigma^n/R^n, \qquad z \text{ on } C_\sigma.$$

Then we have for z on C_σ,

$$| f(z) - p_n(z) | \leqq \frac{\sigma^n}{R^n}\left[M' + 2M \frac{R_n^n Q_n^{(\sigma)}(z)}{\sigma^n} \right],$$

from which one can easily prove*

$$\overline{\lim_{n\to\infty}} \, [\max | f(z) - p_n(z) |, z \text{ on } C_\sigma]^{1/n} \leqq \sigma/R.$$

That is to say, we have proved maximal convergence of the sequence $p_n(z)$ to $f(z)$ on every C_σ, $\sigma > \tau$, from which the theorem follows.

§7.10. Least squares and interpolation in roots of unity

We have indicated a close connection (Theorem 7) between interpolation in roots of unity and Tchebycheff approximation to *analytic* functions. There is a

* The relation $\rho_n^{1/n} \to 1$, $\rho_n > 0$, implies $(\rho_n + k)^{1/n} \to 1$, if $k > 0$, for we have $\rho_n + k > \rho_n$, and we have $\rho_n + k \leqq 2k$ if $\rho_n \leqq k$, $\rho_n + k \leqq 2\rho_n$ if $\rho_n \geqq k$.

connection no less close between interpolation in roots of unity and approximation in the sense of least squares. For the case of analytic functions, this connection appears from Theorem 1; for the case of functions not necessarily analytic we shall prove [Walsh, 1932b]

THEOREM 10. *Let $F(z)$ be an arbitrary function defined and continuous (or more generally integrable in the sense of Riemann) merely on the circumference $C: |z| = 1$. Let $p_n(z)$ be the polynomial in z of degree n of interpolation to $F(z)$ in the $(n + 1)$-st roots of unity. Then we have*

$$\lim_{n\to\infty} p_n(z) = f(z) \equiv \frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t - z}, \qquad |z| < 1,$$

and this limit is approached uniformly on any closed point set interior to C. The function $f(z)$ is the function of class H_2 which is the limit (§6.11, Theorem 15) *interior to C of the formal expansion of $F(z)$ on C in terms of the orthogonal set $1, z, z^2, \cdots$.*

A special case of this theorem is the illustration due to Méray (§3.2), $F(z) \equiv 1/z$, $p_n(z) \equiv z^n$, $\lim_{n\to\infty} p_n(z) \equiv 0$, for $|z| < 1$. If we similarly choose $F(z) = 1/z^k$ $(k > 0)$, we have $p_n(z) = z^{n-k+1}$, $n > k$, for we verify at once the equality of $1/z^k$ and z^{n-k+1} in any point z such that $z^{n+1} = 1$. Theorem 10 is immediate, then, whenever $F(z)$ is of the form $1/z^k$, $k > 0$. Theorem 10 in the general case seems plausible, by the expression (§6.11, Theorem 16) of $F(z)$ as the sum of a function of class H_2 and a function of class G_2; the conclusion of Theorem 10 has already been proved for typical functions (e.g., powers of z) of class H_2 and of class G_2, hence holds for any linear combination of such functions. Our formal proof is independent of these remarks

If we introduce the notation $\omega = e^{2\pi i/(n+1)}$, Lagrange's interpolation formula (§3.1) yields

$$p_n(z) = \sum_{k=1}^{n+1} F(\omega^k)\,\frac{\omega^k(z^{n+1} - 1)}{(n + 1)\,(z - \omega^k)} \tag{48}$$

With the exception of the term z^{n+1} in the numerator, which approaches zero, equation (48) suggests computation of the integral which defines $f(z)$, by division of the circle C at the points ω^k. We have

$$f(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\sum_{k=1}^{n+1} \frac{F(\omega^k)\,(\omega^{k+1} - \omega^k)}{\omega^k - z}, \qquad |z| < 1, \tag{49}$$

$$\lim_{n\to\infty} [f(z) - p_n(z)] = \lim_{n\to\infty}\left[\frac{1}{2\pi i} + \frac{z^{n+1} - 1}{(n + 1)\,(\omega - 1)}\right]\sum_{k=1}^{n+1} \frac{\omega^k(\omega - 1)\,F(\omega^k)}{\omega^k - z}. \tag{50}$$

By (49), the summation in the right-hand member of (50) approaches the limit $2\pi i\, f(z)$, which is continuous for $|z| < 1$, and the limit is approached

uniformly (compare §1.4) for $|z| \leqq r < 1$. The quantity $(n + 1)(\omega - 1)$ approaches $2\pi i$ as its limit, for we have

$$\omega = \cos [2\pi/(n + 1)] + i \sin [2\pi/(n + 1)],$$

$$(51) \qquad \frac{(n + 1)(\omega - 1)}{2\pi i} = \frac{\cos [2\pi/(n + 1)] - 1}{2\pi i/(n + 1)} + \frac{\sin [2\pi/(n + 1)]}{2\pi/(n + 1)};$$

the last member approaches the limit unity. The square bracket in the right-hand member of (50) thus approaches zero for $|z| < 1$, uniformly for $|z| \leqq r < 1$, and the factor of this bracket is bounded uniformly in z and n, $|z| \leqq r < 1$, so the proof is complete.

If the function $F(z)$ is analytic for $|z| < 1$, continuous for $|z| \leqq 1$, it is clear that $f(z)$ coincides with $F(z)$. For this special case, Theorem 10 is due to Fejér [1918].

A generalization of Theorem 10 to the case where C is more general than the unit circle has recently been given by Curtiss [1935].

Of further interest is the

COROLLARY. *Let $F(z)$ be an arbitrary function defined and continuous (or more generally integrable in the sense of Riemann) on the circumference C: $|z| = 1$. Let $p_{-n}(z)$ be the polynomial in $1/z$ of degree n which vanishes at infinity and which interpolates to $F(z)$ in the n-th roots of unity. Then we have*

$$\lim_{n\to\infty} p_{-n}(z) = g(z) \equiv \frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t - z}, \qquad |z| > 1,$$

where the integral is taken in the clockwise sense, and this limit is approached uniformly on any closed limited point set exterior to C. The function $g(z)$ is the function of class G_2 which is the limit exterior to C of the formal expansion of $F(z)$ on C in terms of the orthogonal set $z^{-1}, z^{-2}, \cdots$.

This Corollary can be proved either by following the method used in Theorem 10, or by using the transformation $t' = 1/t$, $z' = 1/z$ and applying Theorem 10.

Theorem 10 and its Corollary are analogous to the (formal) Taylor development of a function. There is a corresponding analogue [Walsh, 1933] of the Laurent development, and which involves these same functions $f(z)$ and $g(z)$:

THEOREM 11. *Let C be the unit circle, and let $F(z)$ be continuous (or more generally integrable in the sense of Riemann) on C. Let $Q_{2n}(z)$ be the polynomial in z and $1/z$ of the form*

$$a_{n,-n}z^{-n} + a_{n,-n+1}z^{-n+1} + \cdots + a_{n0} + a_{n1}z + \cdots + a_{nn}z^n$$

which interpolates to $F(z)$ in the $(2n + 1)$-st roots of unity, and let us set

$$q_n(z) = a_{n0} + a_{n1}z + \cdots + a_{nn}z^n, \quad q_{-n}(z) = a_{n,-1}z^{-1} + a_{n,-2}z^{-2} + \cdots + a_{n,-n}z^{-n}$$

Then we have

$$(52)\qquad \begin{aligned} \lim_{n\to\infty} q_n(z) &= f(z) \equiv \frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t-z}, && |z| < 1,\\ \lim_{n\to\infty} q_{-n}(z) &= g(z) \equiv \frac{1}{2\pi i}\int_C \frac{F(t)\,dt}{t-z}, && |z| > 1, \end{aligned}$$

the latter integral being taken in the clockwise sense, and these limits are uniform for $|z| \leqq r < 1$ *and* $|z| \geqq 1/r > 1$ *respectively. The functions* $f(z)$ *and* $g(z)$ *of* (52) *are also the functions* $f(z)$ *and* $g(z)$ *of* §6.11, *Theorem* 16.

If $F(z)$ *is analytic in the annulus* $1/\rho < |z| < \rho > 1$, *equations* (52) *are valid respectively for* $|z| < \rho$, *uniformly for* $|z| \leqq R < \rho$, *and for* $|z| > 1/\rho$, *uniformly for* $|z| \geqq 1/R > 1/\rho$. *The polynomial* $Q_{2n}(z)$ *converges to* $F(z)$ *for* $1/\rho < |z| < \rho$, *uniformly for* $1/R \leqq |z| \leqq R < \rho$.

We set here $\omega = e^{2\pi i/(2n+1)}$. Since $Q_{2n}(z)$ is found by interpolation to $F(z)$ in the $2n + 1$ distinct points ω^k, the function $z^n Q_{2n}(z)$ is a polynomial in z of degree $2n$ found by interpolation to $z^n F(z)$ in those points and is given by Lagrange's interpolation formula:

$$z^n Q_{2n}(z) = \sum_{k=1}^{2n+1} \omega^{kn} F(\omega^k)\, \frac{\omega^k(z^{2n+1} - 1)}{(2n+1)(z-\omega^k)},$$

$$Q_{2n}(z) = \frac{z^{2n+1} - 1}{(2n+1)z^n} \sum_{k=1}^{2n+1} \frac{F(\omega^k)\,\omega^{k(n+1)}}{z - \omega^k}.$$

We determine $q_n(z)$ by omitting the negative powers of z. We find

$$(z^{2n+1} - 1)/z^n(z - \omega^k) = z^n + \omega^k z^{n-1} + \omega^{2k} z^{n-2} + \cdots + \omega^{2nk} z^{-n},$$

from which the non-negative powers of z are

$$z^n + \omega^k z^{n-1} + \cdots + \omega^{kn} = (z^{n+1} - \omega^{k(n+1)})/(z - \omega^k).$$

This yields

$$q_n(z) = \sum_{k=1}^{n+1} \frac{F(\omega^k)\,\omega^{k(n+1)}\,(z^{n+1} - \omega^{k(n+1)})}{(2n+1)(z - \omega^k)}.$$

This equation suggests the direct computation of the definite integral in (52), where C is subdivided by the points ω^k:

$$(53)\qquad f(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \sum_{k=1}^{2n+1} \frac{F(\omega^k)(\omega^{k+1} - \omega^k)}{\omega^k - z}, \qquad |z| < 1,$$

and the limit exists uniformly (compare §1.4) for $|z| \leqq r < 1$.

We may now write

$$(54)\qquad \begin{aligned} &\lim_{n\to\infty} [f(z) - q_n(z)]\\ &= \lim_{n\to\infty} \sum_{k=1}^{2n+1} \left[\frac{1}{2\pi i} + \frac{\omega^{nk} z^{n+1} - 1}{(2n+1)\,(\omega - 1)}\right] \frac{F(\omega^k)\,(\omega^{k+1} - \omega^k)}{\omega^k - z}, \quad |z| < 1. \end{aligned}$$

The quantity $(2n + 1)(\omega - 1)$ approaches the limit $2\pi i$, by equations (51). By the uniformity of the convergence in (53) as applied to (54), it follows that the right-hand member of (54) approaches zero uniformly for $|z| \leqq r < 1$, as we were to prove.

The second of equations (52) can be proved in precisely similar manner, as the reader will verify. We proceed to prove the last part of the theorem.

It is now convenient to use the Hermite-Lagrange formula. The function $z^n Q_{2n}(z)$ is a polynomial in z found by interpolation to the function $z^n F(z)$ in the points $z^{2n+1} = 1$. If Γ denotes two contours bounding a closed annular region D containing C in its interior but containing no singular point of $F(z)$, we have

$$z^n F(z) - z^n Q_{2n}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(z^{2n+1} - 1)\, t^n F(t)\, dt}{(t^{2n+1} - 1)\,(t - z)}, \qquad z \text{ interior to } D,$$

$$Q_{2n}(z) = \frac{1}{2\pi i z^n}\int_\Gamma \left[\frac{1}{t - z} - \frac{z^{2n+1} - 1}{(t^{2n+1} - 1)\,(t - z)}\right] t^n F(t)\, dt, \qquad z \text{ arbitrary}.$$

If we omit the negative powers of z we obtain $q_n(z)$:

$$q_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{t^n\,(t^{n+1} - z^{n+1})\, F(t)\, dt}{(t^{2n+1} - 1)\,(t - z)}, \qquad z \text{ arbitrary}.$$

Let Γ' and Γ'' be the respective circles $|z| = R_1$, $1 < R_1 < \rho$; $|z| = 1/R_1$. We can then write

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma'} \frac{F(t)\, dt}{t - z}, \qquad z \text{ interior to } \Gamma',$$

$$(55) \qquad f(z) - q_n(z) = \frac{1}{2\pi i}\int_{\Gamma'} \left[\frac{1}{t - z} - \frac{t^n\,(t^{n+1} - z^{n+1})}{(t^{2n+1} - 1)\,(t - z)}\right] F(t)\, dt$$
$$- \frac{1}{2\pi i}\int_{\Gamma''} \frac{t^n\,(t^{n+1} - z^{n+1})\, F(t)\, dt}{(t^{2n+1} - 1)\,(t - z)}, \qquad z \text{ interior to } \Gamma',$$

where the integral over Γ'' is taken in the clockwise sense. The square bracket in the first of the integrals in (55) can be written

$$(t^n z^{n+1} - 1)/[(t^{2n+1} - 1)\,(t - z)], \qquad |t| = R_1,$$

so the first integral approaches zero uniformly for $|z| \leqq R < R_1$. The second integral in (55) also approaches zero uniformly for $|z| \leqq R < R_1$, $|t| = 1/R_1$, so we have established the first of equations (52) for $|z| < \rho$, uniformly for $|z| \leqq R < \rho$. In particular, if $F(z)$ has a singularity on the circle $|z| = \rho$, the sequence $q_n(z)$ converges maximally to $f(z)$ on and within C.

The remainder of the theorem can be proved in precisely the same way, and is left to the reader.

The function $Q_{2n}(z)$ is of course a trigonometric polynomial on C, and is real on C if $F(z)$ is real on C.

So far as the writer is aware, Theorem 11 is the first known example of a

sequence found by interpolation to an arbitrary function analytic in a given multiply connected region which represents that function in the given region.

One might suspect by analogy with Theorem 1 that the sequence $q_n(z)$ always converges to $f(z)$ maximally for z on C. This is true in case $f(z)$ has a singularity on the circle $|z| = \rho$, as follows from (55), but it need not be true if $f(z)$ has no singularity on the circle $|z| = \rho$, as we illustrate by the example $F(z) = 1/(t - z)$, $0 < t < 1$. The following equations can be verified directly [or from (55)]:

$$\frac{z^n}{t - z} - z^n Q_{2n}(z) = \frac{t^n(z^{2n+1} - 1)}{(t^{2n+1} - 1)(t - z)}, \quad Q_{2n}(z) = \frac{(t^{2n+1} - 1)z^n - t^n(z^{2n+1} - 1)}{(t^{2n+1} - 1)(t - z)z^n},$$

$$f(z) - q_n(z) = -q_n(z) = \frac{t^n(z^{n+1} - t^{n+1})}{(t^{2n+1} - 1)(t - z)},$$

so that $f(z) - q_n(z)$ approaches zero if and only if we have $|z| < \rho = 1/t$.

Even though the relation between interpolation in roots of unity and least squares is as close as Theorems 10 and 11 indicate, the functions represented need no longer be the same if approximating polynomials are required to satisfy auxiliary conditions of interpolation. We shall illustrate this in §11.6.

The present chapter does not by any means include all interesting results on interpolation by polynomials. For instance, let C be a Jordan curve containing the origin $z = 0$ in its interior, and let $w = \Phi(z)$, $z = \Psi(w)$ map the interior of C onto the interior of γ: $|w| = 1$ so that the points $z = 0$ and $w = 0$ correspond to each other. A function $f(z)$ analytic interior to C corresponds to a function $f[\Psi(w)]$ analytic interior to γ, and this new function can be expanded interior to γ in powers of w. That is to say, the function $f(z)$ can be expanded interior to C in a series in terms of the functions $1, \Phi(z), [\Phi(z)]^2, [\Phi(z)]^3, \cdots$. The series takes the form

$$f(z) = a_0 + a_1 z\phi_1(z) + a_2 z^2 \phi_2(z) + a_3 z^3 \phi_3(z) + \cdots,$$

where the functions $\phi_k(z)$ are analytic and different from zero interior to C. The functions $\phi_k(z)$ may themselves be polynomials; more generally they can be replaced by suitable polynomials $p_k(z)$ approximating to them, in such a way that an arbitrary function $f(z)$ analytic interior to C can still be expanded interior to C in the corresponding series of polynomials

$$f(z) = b_0 + b_1 z p_1(z) + b_2 z^2 p_2(z) + b_3 z^3 p_3(z) + \cdots,$$

convergent uniformly on any closed set interior to C. This new series is a series of interpolation; compare §3.2. If C is an analytic Jordan curve, this series converges uniformly on and within C whenever $f(z)$ is analytic on and within C; the polynomials $z^k p_k(z)$ belong to the closed interior of C in the sense of §6.5. For the details concerning the functions $p_k(z)$, the reader may refer to Walsh [1924, 1928, 1929] and Heuser [1934].

CHAPTER VIII

INTERPOLATION BY RATIONAL FUNCTIONS

§8.1. Interpolation formulas

Hitherto we have studied primarily interpolation and approximation by *polynomials* in the complex variable z. We shall now commence the study of interpolation and approximation by more general rational functions, in the extended plane. This new study involves more resources than the previous one, so the results are correspondingly more diverse as well as more general. Our first problem is that of interpolation in prescribed points to a given function by a rational function whose poles are given.

THEOREM 1. *Let the points $\alpha_1, \alpha_2, \cdots, \alpha_n$, not necessarily all distinct, be given. Let also points $\beta_1, \beta_2, \cdots, \beta_{n+1}$ distinct from the α_k but not necessarily distinct from each other, and values $\mu_1, \mu_2, \cdots, \mu_{n+1}$ be given. Then there exists a unique rational function $r(z)$ of the form*

$$\frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)} \tag{1}$$

which takes on the values μ_ν in the points β_ν.

Theorem 1 can be proved by the method used for §3.1, Theorem 1. That is to say, the determination of $r(z)$ depends (even if the β_k are not all distinct) on the solution of a system of $n + 1$ linear equations for the $n + 1$ unknowns b_ν. The vanishing of the determinant Δ of the system is a necessary and sufficient condition that the corresponding *homogeneous* system of $n + 1$ equations should admit a solution $b_0, b_1, \cdots, b_n$ of numbers not all zero, or in other words is a necessary and sufficient condition that there exist a rational function of form (1) not identically zero which vanishes in the $n + 1$ points (distinct or not) β_k. The latter eventuality cannot arise, so Δ is different from zero, the coefficients b_ν exist and are uniquely determined, and the theorem is established.

In Theorem 1 and in the sequel, the possibility that one or more points α_k may be infinite is not excluded. The actual form of (1) breaks down, but the proof remains valid. The proper form both here and below is obtained simply by omitting the factors $z - \alpha_k$ which correspond to infinite values α_k. Similarly, if no α_k is infinite, one or more of the points β_k may be infinite. If precisely ν points β_k are infinite, the requirement of interpolation at infinity is that (1) should be analytic at infinity, and that the first ν terms of its development in negative powers of z should be equal to ν prescribed terms. If one or more points β_k are infinite, the proof of Theorem 1 already given requires slight modification,

for instance by first using a suitable linear transformation under which all the original β_k correspond to finite points of the plane; the theorem remains true.

Any function of form (1) or of form (1) with one or more factors omitted in the denominator, is called a *rational function of degree n*, even if the numerator and denominator have common factors. Then under the conventions already mentioned, Theorem 1 is concerned with an arbitrary rational function of degree n whose poles lie in the points α_ν, and that theorem asserts the possibility of interpolation to arbitrary values in $n + 1$ arbitrary points of the extended plane distinct from the α_ν. Of course the interpolating function $r(z)$ need not actually have poles in all or any of the points α_ν.

We shall continue to make the restriction $\alpha_\nu \neq \beta_k$, rather as a convenience than a necessity. If the function (1) is required to take on a given value say in the point α_1, a finite α_k of multiplicity m, the function (1) cannot actually have a pole in the point α_1, and m factors $z - \alpha_1$ must be common to numerator and denominator. The problem reduces to one of interpolation in points β_ν, including the point α_1, by a rational function which has no pole (formal or otherwise) in α_1.

The system of linear equations considered in connection with Theorem 1 leads directly to an interpolation formula. The natural generalization of Lagrange's formula (§3.1, equation (3)) is

$$r(z) = \sum_{k=1}^{n+1} \mu_k \frac{\omega(z)}{(z - \beta_k)\,\omega'(\beta_k)}, \qquad \omega(z) \equiv \frac{(z - \beta_1)(z - \beta_2)\cdots(z - \beta_{n+1})}{(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)}.$$

This formula is valid whenever the β_k are finite and distinct; the formula admits of direct verification as in §3.1.

In order to find an interpolation formula for the rational function $r(z)$ of form (1) which interpolates *to a given analytic function* $f(z)$ in the points β_ν (instead of merely taking on given values μ_ν), it is convenient first to solve the problem for the particular function $f(z) \equiv 1/(t - z)$, $t \neq \alpha_i$. The difference $f(z) - r(z)$ is then a rational function of z of degree $n + 1$ whose poles lie in the points $\alpha_1, \alpha_2, \cdots, \alpha_n, t$, and which vanishes in the points β_ν; we can write

$$f(z) - r(z) = \kappa \frac{(z - \beta_1)(z - \beta_2)\cdots(z - \beta_{n+1})}{(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)(t - z)}, \tag{2}$$

where κ remains to be determined. We can determine κ by multiplying each term of (2) by $t - z$ and then setting $z = t$. We have

$$1 = \kappa \frac{(t - \beta_1)(t - \beta_2)\cdots(t - \beta_{n+1})}{(t - \alpha_1)(t - \alpha_2)\cdots(t - o_n)},$$

$$\begin{aligned}\frac{1}{t - z} - r(z) \\ = \frac{(z - \beta_1)(z - \beta_2)\cdots(z - \beta_{n+1})(t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_n)}{(t - \beta_1)(t - \beta_2)\cdots(t - \beta_{n+1})(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)(t - z)}.\end{aligned} \tag{3}$$

The validity of equation (3) can be verified directly, even if $t = \alpha_j$. The derivation of (3) is essentially valid, and indeed (3) can be verified directly also in the case that poles of $r(z)$ or points of interpolation (but not both) lie at infinity. Of course the points β_j or α_k need not be all distinct, but we assume $\beta_j \neq \alpha_k$.

An interpolation formula for interpolation to an arbitrary analytic function $f(z)$ can now be obtained (compare §3.1) by multiplying (3) through by the factor $f(t)dt/(2\pi i)$ and integrating over the boundary of one or more closed regions in which $f(z)$ is analytic and which contain the points β_ν.

THEOREM 2. *Let C be a closed limited region or several closed limited regions whose boundary Γ consists of a finite number of non-intersecting rectifiable Jordan curves, let the points $\beta_1, \beta_2, \cdots, \beta_{n+1}$ lie interior to C, and let the function $f(z)$ be analytic in C. If $r(z)$ denotes the rational function of degree n whose poles lie in the points $\alpha_1, \alpha_2, \cdots, \alpha_n$, and which interpolates to $f(z)$ in the points β_ν distinct from the α_k, then we have*

$$(4)\quad f(z) - r(z) =$$
$$\frac{1}{2\pi i}\int_\Gamma \frac{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_{n+1})(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)\,f(t)\,dt}{(t-\beta_1)(t-\beta_2)\cdots(t-\beta_{n+1})(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)(t-z)},$$
$$z \text{ interior to } C,\quad z \neq \alpha_k,$$

$$(5)\quad r(z) = \frac{1}{2\pi i}\int_\Gamma\left[1 - \frac{(z-\beta_1)\cdots(z-\beta_{n+1})(t-\alpha_1)\cdots(t-\alpha_n)}{(t-\beta_1)\cdots(t-\beta_{n+1})(z-\alpha_1)\cdots(z-\alpha_n)}\right]\frac{f(t)\,dt}{t-z},$$
$$z \neq \alpha_k.$$

To be sure, equation (5) is not literally valid for all $z \neq \alpha_k$ because the integrand is not defined when z equals t, hence the integral is not defined for values of z on Γ. Nevertheless, the integrand considered as a function of z has only an artificial singularity at such a point of Γ. If the integrand of (5) is replaced by its limit whenever that integrand fails to be defined, then equation (5) has a meaning and is valid for all $z \neq \alpha_k$. If (5) is valid for z interior to C, it is also valid for all $z \neq \alpha_k$, for the integrand now has no singularity in t when z lies on Γ.

In both (4) and (5) integration is to be taken in the positive sense with respect to the region or regions C. Then (4) and (5) are equivalent, by virtue of Cauchy's integral formula for the function $f(z)$ and the region or regions C.

We can verify at once from (4) that the function $r(z)$ defined by (5) interpolates to $f(z)$ in the points β_ν, for equation (4) is valid for z interior to C, hence is valid in the neighborhoods of the points β_ν, and in such neighborhoods $f(z) - r(z)$ is the product of the polynomial $(z-\beta_1)(z-\beta_2)\cdots(z-\beta_{n+1})$ and an analytic function. It follows from (5) that $r(z)$ as defined by (5) is a rational function of z of degree n whose poles lie in the points α_k. The conditions of interpolation and of location of poles determine $r(z)$ uniquely, by Theorem 1, so Theorem 2 is established. Points α_k may lie in C, and of course points α_k may be infinite; in the latter case the factors containing α_k are simply to be omitted.

If we attempt to use equation (4) when the region (or set of regions) C is infinite, no difficulty is experienced provided $f(z)$ is analytic in C. The integrand of (4) has a zero in t at infinity of order at least two, so the equation is valid without change. That is to say, in the usual form of Cauchy's integral applied to an infinite region the integral is taken around the boundary of the region, and an additional constant term is added which is (except for the factor $2\pi i$) merely the coefficient of t^{-1} in the expansion at infinity in negative powers of t of the integrand. Under the present circumstances the additional term vanishes in (4) but need not vanish in (5). The previous verification of (4) and (5) is valid if we add this possible constant term in (5).

Moreover, if C is infinite one or more of the points β_ν may be infinite. This fact is expressed in (4) simply by omitting the corresponding factors, as one can see by comparison with (2). Equation (4) is also valid in this case, as appears by inspection.

Corollary. *Equation* (4) *is valid even if C is not limited and even if points β_ν are infinite.*

Theorem 2 together with its Corollary includes general results on interpolation by functions of the form

$$\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)},$$

in $m + 1$ prescribed points β_k. The case $m > n$ is clearly included, thanks to our convention regarding points α_k which are infinite. In considering the case $m < n$, we may assume no point α_k infinite. Then from the prescribed form of the function, it follows that the rational function (if not identically zero) has a zero of order at least $n - m$ at infinity. Equation (4) is valid in this case, with C assumed finite and with the factors containing $\beta_{m+2}, \beta_{m+3}, \cdots, \beta_{n+1}$ omitted. We interpolate to the given function $f(z)$ in the points β_k interior to C, and at infinity counted $n - m$ times to the function which is identically zero. According to the Corollary we integrate over Γ, the boundary of C, and also form the integral corresponding to (4) where the region corresponding to C is infinite and the function corresponding to $f(t)$ vanishes identically. The latter integral is clearly zero, so (4) is valid in its modified form when C is finite. When C is infinite, equation (4) in its modified form naturally implies interpolation to $f(z)$ at infinity of order $n - m$.

It is of course possible to base the entire study of interpolation not on rational functions of form (1), but on functions say of form

$$\frac{b_0 z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1}}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)}$$

with the usual convention for infinite points α_k. This new form has the advantage over (1) of involving interpolation in n points (that is, the number of points

β_k is the same as the number of points α_k; compare Theorem 3 below) but the new form has the relative disadvantage of representing a rational function which vanishes at infinity, so that an entirely new problem is obtained by linear transformation of the complex variable; moreover, the new form presents an additional complication if some of the β_k lie at infinity. We shall continue to use form (1), mainly because of its invariant character under linear transformation of the complex variable. A certain amount of awkwardness inevitably arises, particularly in §§8.3 and 8.5, for we are studying an invariant problem by the use of Cauchy's integral and related formulas, which are not invariant.

§8.2. Sequences and series of interpolation

We proceed now with the study of convergence of sequences of rational functions with preassigned poles found by interpolation to a given function in given points; the case that all the poles lie at infinity is the case of interpolation by polynomials, which has already been discussed. Our present problem, to be more explicit, can be formulated as

PROBLEM I. *Given the sequences*

$$\begin{array}{l} \alpha_{11}, \\ \alpha_{21}, \alpha_{22}, \\ \alpha_{31}, \alpha_{32}, \alpha_{33}, \\ \cdots\cdots\cdots\cdots, \end{array} \tag{6}$$

$$\begin{array}{l} \beta_{01}, \\ \beta_{11}, \beta_{12}, \\ \beta_{21}, \beta_{22}, \beta_{23}, \\ \cdots\cdots\cdots\cdots, \end{array} \tag{7}$$

with $\beta_{nk} \neq \alpha_{nj}$, and a function $f(z)$ defined in the points β_{nk}. To study the convergence of the sequences of functions of form

$$r_n(z) \equiv \frac{b_{n0}z^n + b_{n1}z^{n-1} + \cdots + b_{nn}}{(z - \alpha_{n1})(z - \alpha_{n2}) \cdots (z - \alpha_{nn})} \tag{8}$$

found by interpolation to $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{n,n+1}$.

In this statement, the requirement that $f(z)$ be defined in the points β_{nk} is intended to include the requirement that suitable derivatives of $f(z)$ exist in multiple points β_{nk}. That condition will be satisfied in our applications by the requirement that $f(z)$ is analytic in the points β_{nk}. The case that points α_{nk} or β_{nk} are infinite is not excluded; the conventions already made (§8.1) apply. Equation (8) is merely intended to require that $r_n(z)$ shall be a rational function of degree n all of whose poles (multiplicities considered) lie in the points α_{nk}.

There are various sorts of hypotheses that it is natural to make on the sequences (6) and (7) (compare §7.1), as will appear in the sequel.

The sequence $r_n(z)$ takes the particularly simple form of the partial sums of a series of interpolation in the case that α_{nk} and β_{nk} do not depend on n, $\alpha_{nk} = \alpha_k$, $\beta_{nk} = \beta_k$. In fact, under these circumstances the function $r_n(z) - r_{n-1}(z)$ is a rational function of degree n which vanishes in the points $\beta_1, \beta_2, \cdots, \beta_n$ because both $r_n(z)$ and $r_{n-1}(z)$ interpolate to $f(z)$ in those points, and whose poles lie in the points $\alpha_1, \alpha_2, \cdots, \alpha_n$. That is to say, $r_n(z) - r_{n-1}(z)$ can be written

$$a_n \frac{(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)},$$

so $r_n(z)$ is the sum of the first n terms of a series of the form

$$a_0 + a_1 \frac{z - \beta_1}{z - \alpha_1} + a_2 \frac{(z - \beta_1)(z - \beta_2)}{(z - \alpha_1)(z - \alpha_2)} + \cdots. \tag{9}$$

In writing (9), we have regard to the usual convention for points α_k or β_ν which are infinite. In particular if all the α_k are infinite the denominators in (9) are to be omitted and (9) takes the form of the right-hand member of §3.2, equation (8). On the other hand, if all the β_ν are infinite, (9) takes the form

$$a_0 + \frac{a_1}{z - \alpha_1} + \frac{a_2}{(z - \alpha_1)(z - \alpha_2)} + \cdots.$$

An arbitrary function $f(z)$ defined in the points β_ν possesses a formal expansion (9) determined by interpolation. We set $z = \beta_1$ and find $a_0 = f(\beta_1)$; set $z = \beta_2$ (if $\beta_2 \neq \beta_1$) and find a_1 from the equation

$$f(\beta_2) = a_0 + a_1 \frac{\beta_2 - \beta_1}{\beta_2 - \alpha_1};$$

and so on. In general, suppose that $a_0, a_1, \cdots, a_{n-1}$ have been determined and that precisely m of the points $\beta_1, \beta_2, \cdots, \beta_n$ are equal to β_{n+1}. Then the m-th derivative of the factor of a_n:

$$\frac{(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)}$$

is different from zero at the point $z = \beta_{n+1}$, whereas the m-th derivatives of the factors of $a_{n+1}, a_{n+2}, \cdots$, all vanish at this point $z = \beta_{n+1}$. Thus the coefficient a_n is uniquely determined and can be computed in terms of $a_1, a_2, \cdots, a_{n-1}$, hence in terms of the α_k, β_ν, and the derivatives of $f(z)$ at the points β_k; compare §3.2. The reader will make the appropriate modifications here for points β_k at infinity.

The formal expansion (9) of $f(z)$ found directly by interpolation to $f(z)$ naturally coincides with the formal expansion of $f(z)$ found first by defining the $r_n(z)$ as in Problem I and then writing the $(n + 1)$-st term of (9) as $r_n(z) - r_{n-1}(z)$; in fact, the sum of the first $n + 1$ terms of the formal expansion (9) found directly

by interpolation to $f(z)$ is a rational function of form (8) which interpolates to $f(z)$ in the points $\beta_1, \beta_2, \cdots, \beta_{n+1}$, hence (Theorem 1) coincides with $r_n(z)$.

A series (9) which is the formal expansion of a function $f(z)$ can be transformed by a linear transformation of the complex variable. Thanks to the conventions already made, series (9) is transformed into a new series of type (9) which interpolates to the transform of $f(z)$ in the points which are the transforms of the β_ν.

A formula for the coefficients a_n in (9) can be obtained at once. We find from (4), under the hypothesis of Theorem 2,

$$r_n(z) - r_{n-1}(z) = a_n \frac{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_n)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}$$
$$= \frac{-1}{2\pi i}\int_\Gamma \left[\frac{(z-\beta_1)\cdots(z-\beta_{n+1})(t-\alpha_1)\cdots(t-\alpha_n)}{(t-\beta_1)\cdots(t-\beta_{n+1})(z-\alpha_1)\cdots(z-\alpha_n)} - \frac{(z-\beta_1)\cdots(z-\beta_n)(t-\alpha_1)\cdots(t-\alpha_{n-1})}{(t-\beta_1)\cdots(t-\beta_n)(z-\alpha_1)\cdots(z-\alpha_{n-1})}\right]\frac{f(t)\,dt}{(t-z)}.$$

Thus we find

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{(\beta_{n+1}-\alpha_n)(t-\alpha_1)\cdots(t-\alpha_{n-1})\,f(t)\,dt}{(t-\beta_1)\cdots(t-\beta_{n+1})}.$$

The equation corresponding to (9), which holds at least in the points β_k,

$$f(z) = a_0 + a_1\frac{z-\beta_1}{z-\alpha_1} + a_2\frac{(z-\beta_1)(z-\beta_2)}{(z-\alpha_1)(z-\alpha_2)} + \cdots, \qquad \alpha_j \neq \beta_k, \tag{10}$$

can be transformed into the equivalent form

$$\frac{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_m)}{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_m)}\left[f(z) - a_0 - a_1\frac{z-\beta_1}{z-\alpha_1} - \cdots - a_{m-1}\frac{(z-\beta_1)\cdots(z-\beta_{m-1})}{(z-\alpha_1)\cdots(z-\alpha_{m-1})}\right] \tag{11}$$
$$= a_m + a_{m+1}\frac{z-\beta_{m+1}}{z-\alpha_{m+1}} + a_{m+2}\frac{(z-\beta_{m+1})(z-\beta_{m+2})}{(z-\alpha_{m+1})(z-\alpha_{m+2})} + \cdots,$$

again an equation precisely of type (10). Equations (10) and (11) are entirely equivalent as series of interpolation, in the sense that if the coefficients $a_0, a_1, \cdots, a_{m-1}$ are determined from (10) by interpolation in the points $z = \beta_1, \beta_2, \cdots, \beta_m$, then determination of the remaining coefficients $a_m, a_{m+1}, \cdots$ from (10) by interpolation in the points $z = \beta_{m+1}, \beta_{m+2}, \cdots$ is equivalent to determination of the coefficients $a_m, a_{m+1}, \cdots$ from (11) by interpolation in the points $z = \beta_{m+1}, \beta_{m+2}, \cdots$. In fact, no matter which of these methods is used to determine the coefficients $a_k (k \geqq m)$, the function

$$a_0 + a_1\frac{z-\beta_1}{z-\alpha_1} + \cdots + a_n\frac{(z-\beta_1)\cdots(z-\beta_n)}{(z-\alpha_1)\cdots(z-\alpha_n)}$$

is the unique rational function of form (1) which interpolates to $f(z)$ in the points $\beta_1, \beta_2, \cdots, \beta_{n+1}$; the a_k are successively uniquely determined.

If the function $f(z)$ is analytic in all the points $\beta_1, \beta_2, \cdots, \beta_m$, so also is the left-hand member of (11), for the square bracket in (11) vanishes in each of the points $\beta_1, \beta_2, \cdots, \beta_m$. Moreover, if $f(z)$ is analytic in the points $\alpha_1, \alpha_2, \cdots, \alpha_m$, so also is the left-hand member of (11). If (11) is valid uniformly on some closed point set containing none of the points $\alpha_1, \alpha_2, \cdots, \alpha_m$, so also is (10), for on such a point set the factor

$$\frac{(z - \beta_1) \cdots (z - \beta_m)}{(z - \alpha_1) \cdots (z - \alpha_m)}$$

is uniformly limited.

Equation (11) has two important advantages over equation (10): (i) in replacing (10) by (11) we study the validity of those equations by the use of the remainder formula (4) where now the first m points of interpolation β_ν and first m poles α_k are omitted, and where $f(z)$ is replaced by the square bracket in (11), so it is clear that the validity of equations (10) and (11) for various values of z and for general $f(z)$ depends primarily on the behavior of the sequences β_ν and α_k for large ν and k; (ii) if $f(z)$ has poles in some or all of the points $\alpha_1, \alpha_2, \cdots, \alpha_m$, and if in this set each point is enumerated at least a number of times corresponding to its order as pole of $f(z)$, then the left-hand member of (11) (when the artificial singularities are removed) is *analytic* in each of the points $\alpha_1, \alpha_2, \cdots, \alpha_m$; that is to say, the expansion of a *meromorphic* function $f(z)$ with poles in points α_k (finite or infinite in number) in the series (10) can be reduced to the problem of expanding a function which is *analytic* wherever $f(z)$ is analytic and also analytic in the points $\alpha_1, \alpha_2, \cdots, \alpha_m$; equation (4) can be used in connection with the new function.

A further consequence of the use of (11) is the possibility of formal expansion of a given function $f(z)$ not originally defined in *all* the given points β_k of (10), for instance not defined in the points $\beta_1, \beta_2, \cdots, \beta_m$. Under these circumstances, arbitrary values may be given to $f(z)$ in those points, or (an equivalent statement) arbitrary values may be chosen for the coefficients $a_0, a_1, \cdots, a_{m-1}$, the coefficients $a_m, a_{m+1}, \cdots$ can still be determined from (11). It may still occur that equations (10) and (11) are valid for suitably chosen z. Similarly, even if $f(z)$ is defined in the points $\beta_1, \beta_2, \cdots, \beta_m$, it may be desirable to modify the definition in those points; so far as (11) is concerned, arbitrary definitions can be given to $f(z)$ in those points, the formal development still exists.

As an application of this remark, we note that a series of form (10) or (11) may represent the function $f(z) \equiv 0$ in a region not containing *all* the points β_k whereas not all the coefficients a_n are zero. For instance, let us choose $\beta_1 = 1$, $\beta_k = 0$ for $k > 1$, $\alpha_k = \infty$, and let $f(z)$ vanish identically in the neighborhood of the origin, whereas $f(1) = 1$. We have $a_0 = 1$, and equation (11) can be written

$$\frac{1}{z-1}[f(z) - 1] = a_1 + a_2 z + a_3 z^2 + \cdots$$
$$= 1 + z + z^2 + \cdots, \qquad |z| < 1.$$

Then (10) takes the form

$$f(z) \equiv 1 + (z-1) + (z-1)z + (z-1)z^2 + \cdots \equiv 0, \qquad |z| < 1.$$

If a series of form (10) converges to $f(z)$ in all the points β_k, and if the β_k are all distinct, then the coefficients a_k must be those of the formal expansion of $f(z)$. If a series of form (10) converges to $f(z)$ in all the points β_k not necessarily distinct, and if the suitable derived series converge to the corresponding derivatives of $f(z)$ in the multiple points β_k (this condition is surely satisfied if the series converges to $f(z)$ uniformly in some neighborhood of each β_k), then that series (10) must be the formal expansion of $f(z)$.

Various expansions similar to (10) are also of importance, such as

$$f(z) = \frac{a_0}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_m)} + \frac{a_1(z-\beta_1)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_{m+1})} + \frac{a_2(z-\beta_1)(z-\beta_2)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_{m+2})} + \cdots,$$

$$f(z) = \frac{a_0(z-\beta_1)\cdots(z-\beta_m)}{z-\alpha_1} + \frac{a_1(z-\beta_1)\cdots(z-\beta_{m+1})}{(z-\alpha_1)(z-\alpha_2)} + \frac{a_2(z-\beta_1)\cdots(z-\beta_{m+2})}{(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)} + \cdots.$$

Such expansions may be treated either (a) by virtue of our general remarks, where suitable points β_k or α_k are chosen at infinity and where suitable coefficients in the general development (10) vanish, or (b) by writing these expansions in the respective equivalent forms

$$(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_m)\,f(z) = a_0 + \frac{a_1(z-\beta_1)}{(z-\alpha_{m+1})} + \frac{a_2(z-\beta_1)(z-\beta_2)}{(z-\alpha_{m+1})(z-\alpha_{m+2})} + \cdots,$$

$$\frac{(z-\alpha_1)\,f(z)}{(z-\beta_1)\,(z-\beta_2)\cdots(z-\beta_m)} = a_0 + \frac{a_1(z-\beta_{m+1})}{z-\alpha_2} + \frac{a_2(z-\beta_{m+1})\,(z-\beta_{m+2})}{(z-\alpha_2)\,(z-\alpha_3)} + \cdots,$$

and using the general results.

The formal expansion (10) of an arbitrary function $f(z)$ defined in the points β_k, with proper reference to multiplicity, must converge to $f(z)$ in the points β_k and may also converge to $f(z)$ in other points z. We turn now to the study of such convergence both for series (10) and under the more general situation of Problem I.

§8.3. Duality: general theorems

A very general theorem, from which can be proved nearly all of our results in connection with Problem I, is the following:

THEOREM 3. *Let C be a closed region or a finite number of closed regions whose boundary Γ consists of a finite number of non-intersecting rectifiable Jordan curves. Let the sequences* (6) *and* (7) *be given, $\beta_{nk} \neq \alpha_{nj}$, β_{nk} interior to C. Let the function $f(z)$ be analytic in C, and let $r_n(z)$ denote the rational function of form* (8) *found by interpolation to $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{n,n+1}$. Then the condition*

$$(12) \qquad \lim_{n\to\infty} \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})\,(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{(t - \beta_{n1}) \cdots (t - \beta_{n,n+1})\,(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} = 0$$

uniformly for t on Γ and for z on the boundary of a closed set C' interior to C containing no point α_{nk} implies

$$(13) \qquad \lim_{n\to\infty} r_n(z) = f(z) \text{ uniformly for } z \text{ on } C'.$$

The proof is immediate by the use of (4), integrating (12) term by term after multiplying by $f(t)/(t - z)$; equation (13) valid uniformly on the boundary of C' implies that equation uniformly on C'. As usual, if points α_{nk} or β_{nk} are infinite, the corresponding factors are to be omitted from (12).

COROLLARY 1. *Under the hypothesis of Theorem 3, let us suppose the expression*

$$(14) \qquad \frac{(z - \beta'_{n1}) \cdots (z - \beta'_{nq})(t - \alpha'_{n1}) \cdots (t - \alpha'_{nq})}{(t - \beta'_{n1}) \cdots (t - \beta'_{nq})(z - \alpha'_{n1}) \cdots (z - \alpha'_{nq})} \cdot \frac{(z - \alpha_{n1}) \cdots (z - \alpha_{nq})(t - \beta_{n1}) \cdots (t - \beta_{nq})}{(t - \alpha_{n1}) \cdots (t - \alpha_{nq})(z - \beta_{n1}) \cdots (z - \beta_{nq})},$$

$$q \text{ independent of } n,$$

uniformly limited for t on Γ and for z on the boundary of C', where the β'_{nk} lie interior to C and where the α'_{nk} lie exterior to C'. Then the rational function of degree n whose poles lie in the points $\alpha'_{n1}, \alpha'_{n2}, \cdots, \alpha'_{nq}, \alpha_{n,q+1}, \alpha_{n,q+2}, \cdots, \alpha_{nn}$ found by interpolation to $f(z)$ in the points $\beta'_{n1}, \beta'_{n2}, \cdots, \beta'_{nq}, \beta_{n,q+1}, \beta_{n,q+2}, \cdots, \beta_{n,n+1}$, converges to $f(z)$ uniformly for z on C'.

The proof is immediate from Theorem 3 itself The expression (14) is simply the product of $2q$ cross-ratios, which are invariant under linear transformation; this invariance is in the present connection and frequently elsewhere convenient in proving the boundedness of (14).

A cross-ratio of the form

$$\frac{(z_1 - z_2)\,(z_3 - z_4)}{(z_2 - z_3)\,(z_4 - z_1)},$$

where the z_k are variable, cannot become infinite unless z_2 and z_3 have a common limit point in the extended plane or unless z_1 and z_4 have a common limit point in the extended plane. We formulate

COROLLARY 2. *The condition on* (14) *in Corollary* 1 *is satisfied provided the points* $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nq}$ *have no limit point on the boundary of* C', *the points* $\beta'_{n1}, \beta'_{n2}, \cdots, \beta'_{nq}$ *have no limit point on* Γ, *the points* $\alpha'_{n1}, \alpha'_{n2}, \cdots, \alpha'_{nq}$ *have no limit point on the boundary of* C', *and the points* $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nq}$ *have no limit point on* Γ.

If the points β'_{nk} are identical with the points β_{nk}, there is no restriction on the β'_{nk}; if the α'_{nk} are identical with the α_{nk}, there is no restriction on the α'_{nk}.

The function which appears in (12) is, except for the factor $1/(t - z)$, the remainder in the expansion of the function $1/(t - z)$ found by interpolation in the points β_{nk}. Equations (4) and (5) are deduced from that expansion with a remainder, so equation (12) is precisely the condition that the expansion of the function $1/(t - z)$ should be valid uniformly in t and z, and the conclusion of Theorem 3 follows from it. All of the sequences of interpolation of the present chapter can be similarly interpreted, not merely as found by interpolation to a given function $f(z)$, but as found by suitable integration from an expansion of the function $1/(t - z)$. Of course it is not surprising that an expansion of $1/(t - z)$ should lead to an expansion of an arbitrary analytic function; compare §6.9. But it is significant that here and below an expansion of $1/(t - z)$ involving interpolation to that function should lead to an expansion of an arbitrary analytic function involving interpolation to that arbitrary function.

Condition (12) is entirely symmetric in t and z, and in the α_{nj} and β_{nk}, except that there are $n + 1$ numbers β_{nk} and only n of the numbers α_{nj}. *Condition* (12) *suggests not merely convergence of a sequence of rational functions found by interpolation in the points* β_{nk} *and with poles in the points* α_{nj}, *but also convergence of a sequence found by interpolation in the points* α_{nj} (*with two additional points* α_{nj} *at infinity; this convention can be lightened*) *with poles in the points* β_{nk}. This is the notion of *duality* [Walsh, 1933d], indicated more explicitly in

THEOREM 4. *Let* Γ *consist of a finite number of non-intersecting rectifiable Jordan curves, bounding a closed region or set of closed regions* G_1 *and let* G_2 *denote the complement of* G_1 *closed by the adjunction of* Γ. *Let* Λ *consist of a finite number of non-intersecting rectifiable Jordan curves interior to* G_1, *bounding a closed region or set of closed regions* L_1 *interior to* G_1, *and let* L_2 *denote the complement of* L_1 *closed by the adjunction of* Λ.

Let the points (6) *lie interior to* L_2 *and let the points* (7) *lie interior to* G_1. *Let the condition*

$$\lim_{n\to\infty} \frac{(z - \beta_{n1}) \cdots (z - \beta_{nn})}{(t - \beta_{n1}) \cdots (t - \beta_{nn})} \frac{(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} = 0 \tag{15}$$

be satisfied uniformly for z *on* Λ *and for* t *on* Γ.

Let the points $\beta_{n,n+1}$ *have no limit point on* Γ. *If* $f(z)$ *is analytic in* G_1, *and if the sequence of rational functions* $r_n(z)$ *of respective degrees* n *whose poles lie in the points* $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ *is defined by interpolation to* $f(z)$ *in the points* $\beta_{n1}, \beta_{n2}, \cdots, \beta_{n,n+1}$, *then the sequence* $r_n(z)$ *converges to* $f(z)$ *uniformly in* L_1.

Let the points $\alpha_{n,n+1}$ have no limit point on Λ. If $g(z)$ is analytic in L_2, and if the sequence of rational functions $s_n(z)$ of respective degrees n whose poles lie in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$ is defined by interpolation to $g(z)$ in the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{n,n+1}$, then the sequence $s_n(z)$ converges to $g(z)$ uniformly in G_2.

The function in (12) is not so favorable for our present use as is the function in (15), for the latter is invariant under linear transformation of the z-plane. In fact, the function in (15) is simply the product of n cross-ratios of the form

$$\frac{(z-\beta_{nk})\,(t-\alpha_{nk})}{(t-\beta_{nk})\,(z-\alpha_{nk})}.$$

The entire configuration with which Theorem 4 is concerned is invariant under linear transformation, except of course that the point at infinity must not lie on

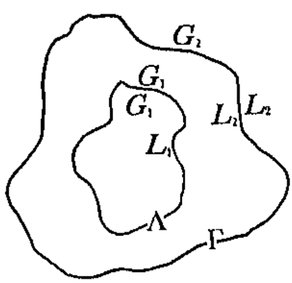

FIG. 2 (diagrammatic)

Γ or Λ, so we may first assume G_1 to be finite. Then for n sufficiently large and for z on Λ and t on Γ the functions

$$(z-\beta_{n,n+1})/(t-\beta_{n,n+1})$$

are uniformly limited, so (15) implies (12) uniformly for z on Λ and t on Γ; the first part of Theorem 4 follows from Theorem 3. In proving the second part of Theorem 4 we assume L_2 to be finite, as we may do. For n sufficiently large and for z on Γ and t on Λ the functions

$$(z-\alpha_{n,n+1})/(t-\alpha_{n,n+1})$$

are uniformly limited, so by (15) we have

$$\lim_{n\to\infty}\frac{(t-\beta_{n1})\cdots(t-\beta_{nn})\,(z-\alpha_{n1})\cdots(z-\alpha_{n,n+1})}{(z-\beta_{n1})\cdots(z-\beta_{nn})\,(t-\alpha_{n1})\cdots(t-\alpha_{n,n+1})}=0$$

uniformly for z on Γ and t on Λ. The second part of Theorem 4 follows as before from Theorem 3.

COROLLARY. *In Theorem 4 the requirement on the limit points of the $\alpha_{n,\,n+1}$ and $\beta_{n,\,n+1}$ may be omitted, provided* (15) *is replaced by the two conditions*

$$\lim_{n\to\infty} \frac{(z-\beta_{n1})\cdots(z-\beta_{n,\,n+1})\,(t-\alpha_{n1})\cdots(t-\alpha_{nn})}{(t-\beta_{n1})\cdots(t-\beta_{n,\,n+1})\,(z-\alpha_{n1})\cdots(z-\alpha_{nn})} = 0\,,$$

uniformly for z on Λ and t on Γ,

$$\lim_{n\to\infty} \frac{(z-\beta_{n1})\cdots(z-\beta_{nn})\,(t-\alpha_{n1})\cdots(t-\alpha_{n,\,n+1})}{(t-\beta_{n1})\cdots(t-\beta_{nn})\,(z-\alpha_{n1})\cdots(z-\alpha_{n,\,n+1})} = 0\,,$$

uniformly for z on Λ and t on Γ.

Most of the results on interpolation that we shall have occasion to consider are contained not merely in Theorem 3 but also in Theorem 4. The latter theorem is entirely symmetric with respect to interchange of the β_{nk} and α_{nk} provided the corresponding regions involved are similarly interchanged. In Theorem 4 the duality is complete.

In Theorems 3 and 4 the points α_{nk}, β_{nk} and functions $r_n(z)$, $s_n(z)$ need not be defined for every n but merely for an infinite sequence of indices n.

Duality varying somewhat in detail occurs in a large number of situations. We now set forth one such situation which occurs frequently in the sequel.

THEOREM 5. *Let R be an annular region bounded by two Jordan curves C_1 and C_2, with C_2 interior to C_1. Let the points $\alpha'_{n1}, \alpha'_{n2}, \cdots, \alpha'_{nn}$ lie on or exterior to C_1 and the points $\beta'_{n1}, \beta'_{n2}, \cdots, \beta'_{nn}$ lie on or interior to C_2 and satisfy the relations*

$$\begin{aligned} &\lim_{n\to\infty} |\,(z-\alpha'_{n1})\,(z-\alpha'_{n2})\cdots(z-\alpha'_{nn})\,|^{1/n} = |\,\Phi_1(z)\,|\,, \\ (16)\qquad &\lim_{n\to\infty} |\,(z-\beta'_{n1})\,(z-\beta'_{n2})\cdots(z-\beta'_{nn})\,|^{1/n} = |\,\Phi_2(z)\,|\,, \end{aligned}$$

uniformly on any closed point set interior to R. Let the function $|\,\Phi(z)\,| \equiv |\,\Phi_2(z)/\Phi_1(z)\,|$ be continuous in the closed region $\bar{R}$, and take constant values γ_1 and $\gamma_2 < \gamma_1$ on C_1 and C_2 respectively. Denote generically by C_γ the curve $|\,\Phi(z)\,| = \gamma$ interior to R, $\gamma_2 < \gamma < \gamma_1$.

If $f(z)$ is analytic on and interior to C_γ, then the sequence of rational functions $r_n(z)$ of respective degrees n with poles in the points $\alpha_{nk} = \alpha'_{nk}$ defined by interpolation to $f(z)$ in the points $\beta_{nk} = \beta'_{n+1,\,k}$ converges to $f(z)$ uniformly on and within C_γ.

If $g(z)$ is analytic on and exterior to C_γ, then the sequence of rational functions $s_n(z)$ of respective degrees n with poles in the points $\beta_{nk} = \beta'_{nk}$ defined by interpolation to $g(z)$ in the points $\alpha_{nk} = \alpha'_{n+1,\,k}$ converges to $g(z)$ uniformly on and exterior to C_γ.

It is consequently true that if $f(z)$ is analytic interior to C_λ but has a singularity on C_λ, then the sequence $r_n(z)$ converges uniformly to $f(z)$ on any closed

set interior to C_λ. If $g(z)$ is analytic exterior to C_λ but has a singularity on C_λ, then the sequence $s_n(z)$ converges uniformly to $g(z)$ on any closed set exterior to C_λ.

The quantities $| z - \alpha'_{nk} |$ and $| z - \beta'_{nk} |$ are uniformly bounded from zero for z on any closed set interior to R, so $\Phi_1(z)$ and $\Phi_2(z)$ cannot vanish in R. Equations (16) imply

$$
(17)\qquad
\begin{aligned}
&\lim_{n\to\infty}\left|\frac{(z-\beta'_{n+1,1})(z-\beta'_{n+1,2})\cdots(z-\beta'_{n+1,n+1})}{(z-\alpha'_{n1})(z-\alpha'_{n2})\cdots(z-\alpha'_{nn})}\right|^{1/n} = |\Phi(z)|,\\
&\lim_{n\to\infty}\left|\frac{(z-\beta'_{n1})(z-\beta'_{n2})\cdots(z-\beta'_{nn})}{(z-\alpha'_{n+1,1})(z-\alpha'_{n+1,2})\cdots(z-\alpha'_{n+1,n+1})}\right|^{1/n} = |\Phi(z)|,
\end{aligned}
$$

uniformly on any closed set interior to R. Conditions (16) and (17) are unchanged if a factor $z - \beta'_{n+1,n+1}$ is removed or inserted, and likewise if a factor $z - \alpha'_{n+1,n+1}$ is removed or inserted, provided $\alpha'_{n+1,n+1}$ is uniformly limited. The exponents $1/n$ may be replaced by $1/(n + 1)$. The case that the α'_{nk} are not uniformly limited is exceptional in (16) and (17) only in form; compare equation (26) and the Corollary to Theorem 8 below.

The proof of Theorem 5 is immediate, for under the hypothesis on $f(z)$ we choose $C_{\gamma'}$, $\gamma' > \gamma$, so that $f(z)$ is analytic on and within $C_{\gamma'}$. The curve $C_{\gamma'}$ is necessarily rectifiable. The curves $C_{\gamma'}$ and C_γ play the respective rôles of the curves Γ and Λ of the Corollary of Theorem 4. By (17) the n-th root of the modulus of the function in the left-hand member of the first equation in that Corollary approaches uniformly for z on C_γ and t on $C_{\gamma'}$ the limit γ/γ', which is less than unity. The first part of Theorem 5 follows, and the second part is similarly proved.

It is of course clear from (17) that for n sufficiently large the curve C_γ must separate $C_{\gamma'}(\gamma' > \gamma)$ from the points β'_{nk}, and that $C_{\gamma'}$ must separate C_γ from the points α'_{nk}, for the function

$$
\left[\frac{(z-\beta'_{n+1,1})\cdots(z-\beta'_{n+1,n+1})}{(z-\alpha'_{n1})\cdots(z-\alpha'_{nn})}\right]^{1/n}
$$

is analytic (although not single-valued) except for branch points and has a single-valued modulus in the entire finite plane except at the points α'_{nk}. When n is large, this function has approximately the modulus γ' on $C_{\gamma'}$. Any Jordan arc connecting a point of $C_{\gamma'}$ to a point β'_{nk} must contain points where the modulus of this function has all values intermediate between γ' and zero; hence any such curve must pass through at least one point of C_γ; this reasoning is developed in more detail and rigor in §4.2. Similarly, any Jordan arc connecting a point of C_γ to a point α'_{nk} must pass through at least one point of $C_{\gamma'}$.

In the hypothesis of Theorem 5 equations (16) can obviously be replaced by equations (17); this adds no generality; compare §9.11, Theorem 18. No generality is added to Theorem 5 if we assume equations (16) or (17) merely to hold uniformly on two Jordan curves, say C'_1: $|\Phi(z)| = \lambda_1 \neq 0$ and C'_2: $|\Phi(z)| =$

$\lambda_2 \neq 0$ both of which separate the β'_{nk} from the α'_{nk}, instead of in a region. Denote by Q the region bounded by C'_1 and C'_2; these two curves cannot intersect. The logarithm of the absolute value which appears in the left-hand member of either equation (17) is harmonic in Q, continuous in the corresponding closed region. This logarithm converges uniformly on the boundary of Q, hence converges uniformly throughout the closed region Q, so equations (17) are valid throughout Q provided $\Phi(z)$ is suitably defined interior to Q.

Theorem 5 was stated in a particularly simple form, but it is obvious that a much more general result can be proved in the same way; the details are left to the reader:

COROLLARY 1. *Let the function $\Phi(z)$ be analytic except possibly for branch points, not necessarily single-valued, but not identically constant interior to a limited region or a finite number of limited regions R. Let $|\Phi(z)|$ be single-valued and continuous on the corresponding closed point set, and take constant values γ_1 and γ_2 on C_1 and C_2, point sets belonging to the boundary of R and together composing the boundary of R. Denote generically by C_γ the locus $|\Phi(z)| = \gamma$, $\gamma_1 > \gamma > \gamma_2$, interior to R, and denote by R'_γ and R''_γ the two sets of regions (mutually complementary with respect to the complement of C_γ respecting the entire plane) into which C_γ separates the plane. Denote by $\bar{R}'_\gamma$ and $\bar{R}''_\gamma$ the sets of closed regions which result from the adjunction of C_γ to R'_γ and R''_γ. Suppose that every R'_γ contains the points β'_{nk} in its interior and that every R''_γ contains the points α'_{nk} in its interior. Suppose finally that equations* (17) *are valid uniformly on any closed set interior to R.*

If $f(z)$ is analytic on $\bar{R}'_\gamma$, then the sequence of rational functions $r_n(z)$ of respective degrees n whose poles lie in the points $\alpha_{nk} = \alpha'_{nk}$ found by interpolation to $f(z)$ in the points $\beta_{nk} = \beta'_{n+1,k}$ converges uniformly to $f(z)$ on $\bar{R}'_\gamma$.

If $g(z)$ is analytic on $\bar{R}''_\gamma$, then the sequence of rational functions $s_n(z)$ of respective degrees n whose poles lie in the points $\beta_{nk} = \beta'_{nk}$ found by interpolation to $g(z)$ in the points $\alpha_{nk} = \alpha'_{n+1,k}$ converges uniformly to $g(z)$ on $\bar{R}''_\gamma$.

We shall study in §§8.7 and 8.8 in some detail the existence of sequences α'_{nk} and β'_{nk} such that (16) and (17) are valid. In both Theorem 5 and Corollary 1, the points α'_{nk} and β'_{nk} and the functions $r_n(z)$ and $s_n(z)$ need not be defined for *every* n, but merely for an infinite sequence of indices n.

The following is essentially included in the proof already suggested:

COROLLARY 2. *Under the conditions of Corollary* 1, *we have*

$$\overline{\lim_{n\to\infty}} \, [\max |f(z) - r_n(z)|, \quad z \text{ on } \bar{R}'_\lambda]^{1/n} \leqq \lambda/\gamma, \qquad \lambda < \gamma,$$

$$\overline{\lim_{n\to\infty}} \, [\max |g(z) - s_n(z)|, \quad z \text{ on } \bar{R}''_\lambda]^{1/n} \leqq \gamma/\lambda, \qquad \lambda > \gamma.$$

It is worth pointing out that under the conditions of Corollary 1, the sequence $r_n(z)$ may actually diverge at every point which is interior to both R and R''_λ, if $f(z)$ is analytic interior to R'_λ but has a singularity on C_λ. We set $f(z) \equiv 1/(t - z)$, whence

$$f(z) - r_n(z) = \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})(t - \beta_{n1}) \cdots (t - \beta_{n,n+1})(t - z)}.$$

If z lies on the locus $|\Phi(z)| = \lambda' > \lambda$, this right-hand member has $\lambda'/\lambda > 1$ for the limit of the n-th root of its modulus, so the sequence $r_n(z)$ diverges.

A somewhat less restrictive condition than (17) is sometimes of importance in connection with Theorem 4. It is clear that equation (15) is satisfied provided we have uniformly (compare §9.13)

$$\overline{\lim_{n\to\infty}} \left| \frac{(z - \beta_{n1}) \cdots (z - \beta_{nn})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \right|^{1/n} \leqq |\Psi_1(z)|, \quad z \text{ on } \Lambda: |\Psi_1(z)| = \lambda,$$

$$\underline{\lim_{n\to\infty}} \left| \frac{(z - \beta_{n1}) \cdots (z - \beta_{nn})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \right|^{1/n} \geqq |\Psi_2(z)|, \quad z \text{ on } \Gamma: |\Psi_2(z)| = \gamma > \lambda;$$

in this case the conclusion of Theorem 4 can be sharpened by writing

$$\overline{\lim_{n\to\infty}} \, [\max |f(z) - r_n(z)|, \quad z \text{ in } L_1]^{1/n} \leqq \lambda/\gamma,$$

$$\overline{\lim_{n\to\infty}} \, [\max |g(z) - s_n(z)|, \quad z \text{ in } G_2]^{1/n} \leqq \lambda/\gamma.$$

A large part of the material of the present chapter can be built into a unified structure with this remark as fundamental.

§8.4. Duality: illustrations

Let us now indicate some configurations [Walsh, 1933d] which are included under Theorems 3, 4, and 5 with their Corollaries. In every case we mention a particular configuration with special properties. A more general configuration is obtained by modifying the sequences α_{nk} and β_{nk} without changing the asymptotic properties, in the sense of (16), and also by transforming by means of an arbitrary linear transformation of the complex variable. The reader can easily formulate explicitly these more general theorems.

Ia. *If the function $f(z)$ is analytic for $|z| \leqq R$, then the sequence of rational functions of respective degrees n with poles at infinity (i.e., sequence of polynomials in z of degree n) and which interpolate to $f(z)$ in the origin considered of multiplicity $n + 1$, converges uniformly to $f(z)$ for $|z| \leqq R$.*

Ib. *If the function $g(z)$ is analytic for $|z| \geqq R$, then the sequence of rational functions of respective degrees n with poles at the origin (i.e., sequence of polynomials in $1/z$ of degree n) and which interpolate to $g(z)$ at infinity considered of multiplicity $n + 1$, converges uniformly to $g(z)$ for $|z| \geqq R$.*

Ia simply deals with the Cauchy-Taylor series, and Ib with that of Laurent. Both of these theorems are contained in Theorem 5, for the first of equations (16) is omitted (or more accurately both members are set equal to unity; $\alpha_{nk} = \infty$) and the second of those equations takes the trivial form

$$\lim_{n\to\infty} |z^n|^{1/n} = |z|.$$

IIa. *If the function $f(z)$ is analytic for $|p(z)| \leqq \mu > 0$, $p(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_l)$, then the sequence of rational functions of respective degrees $ln - 1$ with poles at infinity (i.e., sequence of polynomials in z of degree $ln - 1$) and which interpolate to $f(z)$ in the points z_k each considered of multiplicity n, converges uniformly to $f(z)$ for $|p(z)| \leqq \mu$.*

IIb. *If the function $g(z)$ is analytic for $|p(z)| \geqq \mu > 0$, $p(z) \equiv (z - z_1)(z - z_2) \cdots (z - z_l)$, then the sequence of rational functions of respective degrees ln with poles in the points z_k each of multiplicity n, and which interpolate to $g(z)$ at infinity counted of multiplicity $ln + 1$, converges uniformly to $g(z)$ for $|p(z)| \geqq \mu$.*

IIa expresses the result of the Jacobi series (§3.3). The dual, IIb, is remarkable in that the region of convergence of the sequence of rational functions need not be simply connected, whereas the sequence is defined by interpolation in a single point. The function $g(z)$ enters into the expansion only by means of the coefficients of its Laurent expansion about the point at infinity. The second of equations (16) for IIa is equivalent to

$$\lim_{n \to \infty} |[p(z)]^n|^{1/(ln)} = |p(z)|^{1/l},$$

itself of form (17).

IIIa. *Let C be a closed limited point set whose complement is connected and regular, let the points β_{nk} have no limit point exterior to C, and let the relations* (notation of §7.2)

$$(18) \qquad \lim_{n \to \infty} |(z - \beta_{n1})(z - \beta_{n2}) \cdots (z - \beta_{n,n+1})|^{1/(n+1)} = \Delta |\phi(z)|$$

hold uniformly for z on any closed limited point set exterior to C. If $f(z)$ is analytic on and within C_R, then the sequence of rational functions of respective degrees n with poles at infinity (polynomials in z of degree n) which interpolate to $f(z)$ in the points β_{nk} converges uniformly to $f(z)$ on and within C_R.

IIIb. *Let C and the points β_{nk} satisfy the hypothesis of* IIIa. *If the function $g(z)$ is analytic on and exterior to C_R, then the sequence of rational functions of degree $n + 1$ with poles in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{n,n+1}$, and which interpolate to $g(z)$ at infinity considered of multiplicity $n + 2$, converges uniformly to $g(z)$ on and exterior to C_R.*

IIIa is included in §7.2, Theorem 2 and includes IIa; similarly IIIb includes IIb. Under the present circumstances condition (18) is equivalent to (17) in the form of the second of equations (16). IIIb essentially refers to an *arbitrary* region K (the exterior of C) whether of simple or multiple connectivity, and with only very mild restrictions on K. The theorem yields an expansion valid throughout a multiply connected region K of an arbitrary function analytic in K, an expansion defined by the behavior of the function merely at a single point.

IVa. *If the function $f(z)$ is analytic for $|z| \leqq R$, $B < R < A$, then the sequence of rational functions of respective degrees n with poles in the n points $(A^n)^{1/n}$ and which interpolate to $f(z)$ in the $n + 1$ points $(B^{n+1})^{1/(n+1)}$, converges uniformly to $f(z)$ for $|z| \leqq R$.*

IVb. *If the function $g(z)$ is analytic for $|z| \geqq R$, $B < R < A$, then the sequence of rational functions of respective degrees n with poles in the n points $(B^n)^{1/n}$ and which interpolate to $g(z)$ in the $n + 1$ points $(A^{n+1})^{1/(n+1)}$, converges uniformly to $g(z)$ for $|z| \geqq R$.*

The case $A = \infty$ is not excluded, nor is the case $B = 0$. IVa and IVb can be proved easily each from the other by a substitution $w = 1/z$. Conditions (16) may in IVa be written

$$\lim_{n\to\infty} |z^n - A^n|^{1/n} = A, \qquad |z| < A < \infty,$$

$$\lim_{n\to\infty} |z^{n+1} - B^{n+1}|^{1/(n+1)} = |z|, \qquad |z| > B.$$

IVa for $A = \infty$ contains part of §7.1, Theorem 1.

A remark on the behavior of the sequence of IVa for $|z| > A$ is made in §9.4.

Va. *If the function $f(z)$ is analytic for $1/R \leqq |z| \leqq R > 1$, then the sequence of rational functions $r_{2n}(z)$ of respective degrees $2n$ whose poles lie in the origin and the point at infinity each of multiplicity n and which interpolate to $f(z)$ in the $(2n + 1)$-st roots of unity, converges to $f(z)$ uniformly for $1/R \leqq |z| \leqq R$.*

Vb. *If the function $g(z)$ is analytic for $|z| \leqq 1/R$ and for $|z| \geqq R > 1$, then the sequence of rational functions of respective degrees $2n - 1$ whose poles lie in the $(2n - 1)$-st roots of unity and which interpolate to $g(z)$ in the origin and point at infinity each of multiplicity n, converges to $g(z)$ uniformly for $|z| \leqq 1/R$ and for $|z| \geqq R$.*

Va [Walsh, 1933], like IIIa in certain cases, involves interpolation and convergence in a multiply connected region. Theorem Va has already been proved (§7.10, Theorem 11) in a somewhat different connection. In Va, equations (16) take the form

$$\lim_{n\to\infty} |z^n|^{1/(2n)} = |z|^{1/2}, \qquad \lim_{n\to\infty} |z^{2n+1} - 1|^{1/(2n+1)} = \begin{cases} |z|, & |z| > 1, \\ 1, & |z| < 1, \end{cases}$$

$$\Phi(z) = \begin{cases} z^{1/2}, & |z| > 1, \\ z^{-1/2}, & |z| < 1. \end{cases}$$

In Va the function $r_{2n}(z)$ reduces to a trigonometric polynomial of order n on the circle $|z| = 1$.

An expansion somewhat similar in properties to that of Vb but not defined by a sequence of interpolation of the kind considered in §8.2 has been studied by Ketchum [1934].

Va and Vb can obviously be extended so that the interpolating rational functions are defined for *every* degree; it is sufficient to apply Theorem 5 Corollary 1 with α'_{nk} the k-th element of the sequence $0, \infty, 0, \infty, \cdots$, and with the β'_{nk} the n-th roots of unity.

In Ia and Ib we have developments in powers of z and $1/z$ respectively, but we have not mentioned as yet an analogue of the Laurent series for a function $F(z)$ defined and analytic merely in a ring $R_1 < |z| < R_2$. Such a function $F(z)$ can be split (§1.7) into two component functions: $F(z) \equiv f(z) + g(z)$, where $f(z)$ is analytic for $|z| < R_2$, and $g(z)$ is analytic for $|z| > R_1$, zero at infinity. Then by Ia the function $f(z)$ can be expressed as the limit of a sequence $r_n(z)$ valid for $|z| < R_2$, and by Ib the function $g(z)$ can be expressed as the limit of a sequence $s_n(z)$ valid for $|z| > R_1$. Thus the function $F(z)$ is expressed as the limit of the sequence $r_n(z) + s_n(z)$ in the region $R_1 < |z| < R_2$. However, the sequence $r_n(z) + s_n(z)$ is not determined by *interpolation to* $F(z)$; the component sequences are defined by interpolation to $f(z)$ and $g(z)$ respectively. The two sequences $r_n(z)$ and $s_n(z)$ are mutually dual. The expansion of $F(z)$ in the sequence $r_n(z) + s_n(z)$ can of course be obtained from the corresponding expansion of the function $1/(t - z)$; compare §§8.3 and 6.9.

This general remark (Laurent's theorem) relative to the splitting up of $F(z)$ and development of the components has immediate application not merely to all the other results of §8.4, but also to all the results of §8.3, especially Theorems 3, 4, and 5 (with Corollaries). For instance, under the conditions of Theorem 5, let $F(z)$ be analytic in the annular region bounded by $C_{\lambda'}$ and $C_{\lambda''}$, $\lambda' > \lambda''$. Then in that annular region we can write $F(z) \equiv f(z) + g(z)$, where $f(z)$ is analytic throughout the interior of $C_{\lambda'}$ and $g(z)$ is analytic throughout the exterior of $C_{\lambda''}$. The two sequences $r_n(z)$ and $s_n(z)$ of Theorem 5 found by interpolation to $f(z)$ and $g(z)$ respectively converge respectively interior to $C_{\lambda'}$ and exterior to $C_{\lambda''}$, uniformly on any closed sets in those regions. If we choose an arbitrary λ, $\lambda' > \lambda > \lambda''$, we may write

$$f(z) = \frac{1}{2\pi i}\int_{C_\lambda} \frac{F(t)\,dt}{t - z}, \qquad z \text{ interior to } C_\lambda,$$

$$f(z) - r_n(z) = \frac{1}{2\pi i}\int_{C_\lambda} \frac{(z-\beta_{n1})\cdots(z-\beta_{n,n+1})(t-\alpha_{n1})\cdots(t-\alpha_{nn})F(t)\,dt}{(t-\beta_{n1})\cdots(t-\beta_{n,n+1})(z-\alpha_{n1})\cdots(z-\alpha_{nn})(t-z)},$$

$$z \text{ interior to } C_\lambda;$$

the proof is analogous to that of equation (4). A similar equation holds for $g(z) - s_n(z)$.

We do not amplify this study of sequences of the form $r_n(z)$ and $s_n(z)$, for such sequences are not sequences of interpolation to the given function $F(z)$, hence are beyond the scope of this work. The reader will note, however, that interesting sequences $r_n(z)$ and $s_n(z)$ found by interpolation to the components of a given $F(z)$ suggest themselves not merely in connection with the results of §§8.3 and 8.4, but throughout the remainder of the present chapter. Formulas for the

$r_n(z)$ and $s_n(z)$ in terms of the given function $F(z)$, and regions known to be regions of convergence (the boundaries may be points, and the regions entire domains of definition of $F(z)$), present themselves at once.

§8.5. Duality and series of interpolation

Some of the illustrations given in §8.4, such as IIIa and IIIb, may give rise to series of interpolation. We shall now devote some attention to series of interpolation as such, particularly in the light of Theorems 3, 4, and 5.

THEOREM 6. *Let C be a closed region or several closed regions whose boundary Γ consists of a finite number of non-intersecting rectifiable Jordan curves. Let the sequences $\beta_1, \beta_2, \cdots$ and $\alpha_1, \alpha_2, \cdots$ be given, $\alpha_j \neq \beta_k$, all but a finite number of the β_k interior to C. Let no α_k lie on Γ. Let the function $f(z)$ be meromorphic in C, and let all the poles of $f(z)$ in C occur in the sequence $\alpha_1, \alpha_2, \cdots$ at least a number of times corresponding to their multiplicities. Then the condition*

$$\lim_{n\to\infty} \frac{(z-\beta_1)\cdots(z-\beta_{n+1})(t-\alpha_1)\cdots(t-\alpha_n)}{(t-\beta_1)\cdots(t-\beta_{n+1})(z-\alpha_1)\cdots(z-\alpha_n)} = 0 \tag{19}$$

uniformly for t on Γ and for z on some closed set C' interior to C containing no point α_k and with no point β_k on its boundary implies the uniform convergence to $f(z)$ on C' of the formal expansion determined by interpolation in the points β_k:

$$f(z) = a_0 + a_1\frac{z-\beta_1}{z-\alpha_1} + a_2\frac{(z-\beta_1)(z-\beta_2)}{(z-\alpha_1)(z-\alpha_2)} + \cdots. \tag{20}$$

If the function $f(z)$ is not defined in the points β_j that lie exterior to C, arbitrary values may be used for interpolation in such points. The function $f(z)$ can have at most a finite number of poles in C, so the number m can be chosen so large that the left-hand member of (11) is analytic in C, and likewise so large that all the points $\beta_{m+1}, \beta_{m+2}, \cdots$ lie interior to C. Condition (19) implies the condition

$$\lim_{n\to\infty} \frac{(z-\beta_{m+1})\cdots(z-\beta_{n+1})(t-\alpha_{m+1})\cdots(t-\alpha_n)}{(t-\beta_{m+1})\cdots(t-\beta_{n+1})(z-\alpha_{m+1})\cdots(z-\alpha_n)} = 0 \tag{21}$$

uniformly for t on Γ and for z on C', by virtue of the fact that no α_k lies on Γ and that no β_k lies on the boundary of C'. The set C' can contain no points α_k. If points among $\beta_1, \beta_2, \cdots, \beta_m$ lie interior to C', we first establish (21) on a closed set C'' obtained by deleting from C' suitable neighborhoods of those points β_k; condition (19) for t on Γ and z on C'' implies (21) for t on Γ and z on C''; this implies (21) for t on Γ and z on C'. Even if a point α_k, β_ν, or z can be infinite, the factor omitted from (19):

$$\frac{(z-\beta_1)\cdots(z-\beta_m)(t-\alpha_1)\cdots(t-\alpha_m)}{(t-\beta_1)\cdots(t-\beta_m)(z-\alpha_1)\cdots(z-\alpha_m)} \tag{22}$$

is uniformly bounded from zero for z on C'', for (22) is precisely the product of m cross-ratios each of which is uniformly bounded from zero on C''.

Theorem 6 is now a consequence of (21), of Theorem 3, and of the equivalence of (11) and (20) as established in §8.2.

Duality for series of interpolation is just as complete as for sequences of interpolation:

THEOREM 7. *Let Γ consist of a finite number of non-intersecting rectifiable Jordan curves, bounding a closed region or set of closed regions G_1 and let G_2 denote the complement of G_1 closed by the adjunction of Γ. Let Λ consist of a finite number of non-intersecting rectifiable Jordan curves interior to G_1, bounding a closed region or set of closed regions L_1 interior to G_1, and let L_2 denote the complement of L_1 closed by the adjunction of Λ.*

Let all but a finite number of the points $\beta_1, \beta_2, \cdots$ lie in G_1, and let no point β_ν lie on Λ. Let all but a finite number of the points $\alpha_1, \alpha_2, \cdots, \alpha_j \neq \beta_k$, lie in L_2 and let no point α_k lie on Γ. Let the condition

$$\lim_{n\to\infty} \frac{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_n)(t-\alpha_1)\cdots(t-\alpha_n)}{(t-\beta_1)(t-\beta_2)\cdots(t-\beta_n)(z-\alpha_1)\cdots(z-\alpha_n)} = 0 \tag{23}$$

be satisfied uniformly for z on Λ and for t on Γ.

If no limit point of the β_k lies on Γ, if $f(z)$ is meromorphic in G_1, and if all the poles of $f(z)$ in G_1 occur in the sequence α_k (with proper attention to multiplicity), then the formal development (20) *of $f(z)$ converges to $f(z)$ throughout L_1 except at the points α_k, and the convergence is uniform in L_1 except in the neighborhoods of such points α_k as lie in L_1.*

If no limit point of the α_k lies on Λ, if $g(z)$ is meromorphic in L_2, and if all the poles of $g(z)$ in L_2 occur in the sequence β_k, then the formal development of $g(z)$ determined by interpolation in the points α_k,

$$g(z) = b_0 + b_1 \frac{z-\alpha_1}{z-\beta_1} + b_2 \frac{(z-\alpha_1)(z-\alpha_2)}{(z-\beta_1)(z-\beta_2)} + \cdots, \tag{24}$$

converges to $g(z)$ throughout G_2 except at the points β_k, and the convergence is uniform in G_2 except in the neighborhoods of such points β_k as lie in G_2.

The entire situation of Theorem 7 is unchanged by a linear transformation, except that Γ and Λ must be finite, so it is allowable to assume L_1 and G_2 finite regions. For z on L_1 and t on Γ and for n sufficiently large the functions

$$(z-\beta_{n+1})/(t-\beta_{n+1})$$

are uniformly limited, so condition (23) implies (19) uniformly for t on Γ and z on L_1 except in the neighborhood of such points α_k (at most finite in number) as lie in L_1. The first part of Theorem 7 now follows from Theorem 6, and the second part of Theorem 7 is similarly proved. The following corollary follows directly from Theorem 6:

COROLLARY. *In Theorem 7 the requirements on the limit points of the α_k and β_k may be omitted, provided* (23) *is replaced by the two conditions*

$$\lim_{n\to\infty} \frac{(z-\beta_1)\cdots(z-\beta_{n+1})(t-\alpha_1)\cdots(t-\alpha_n)}{(t-\beta_1)\cdots(t-\beta_{n+1})(z-\alpha_1)\cdots(z-\alpha_n)} = 0\,,$$

uniformly for z *on* Λ *and* t *on* Γ ,

$$\lim_{n\to\infty} \frac{(z-\beta_1)\cdots(z-\beta_n)(t-\alpha_1)\cdots(t-\alpha_{n+1})}{(t-\beta_1)\cdots(t-\beta_n)(z-\alpha_1)\cdots(z-\alpha_{n+1})} = 0\,,$$

uniformly for z *on* Λ *and* t *on* Γ .

We prove likewise the analogue of Theorem 5 and its Corollary 1:

THEOREM 8. *Let the function* $\Phi(z)$ *be analytic except possibly for branch points, not necessarily single-valued but not identically constant interior to a limited region or finite number of limited regions* R. *Let* $|\Phi(z)|$ *be single-valued and continuous on the corresponding closed point set, and take constant values* γ_1 *and* $\gamma_2 < \gamma_1$ *on* C_1 *and* C_2, *point sets belonging to the boundary of* R *and together composing the boundary of* R. *Denote generically by* C_γ *the locus* $|\Phi(z)| = \gamma$, $\gamma_1 > \gamma > \gamma_2$, *interior to* R, *and denote by* R'_γ *and* R''_γ *two complementary* (*with respect to the complement of* C_γ) *sets of regions into which* C_γ *separates the plane. Suppose that every* R'_γ *contains in its interior all the limit points of the* β_k, *and that every* R''_γ *contains in its interior all the limit points of the* α_k. *Suppose finally that the relation*

$$\lim_{n\to\infty} \left| \frac{(z-\beta_1)\cdots(z-\beta_n)}{(z-\alpha_1)\cdots(z-\alpha_n)} \right|^{1/n} = |\Phi(z)| \tag{25}$$

holds uniformly on each of a set of curves C_γ *everywhere dense in* R, *with* $\alpha_n \neq \beta_k$.

If $f(z)$ *is meromorphic on the closed set* $\bar{R}'_\gamma$, *and if all of the poles of* $f(z)$ *in this closed set occur in the sequence* α_k, *then the formal development* (20) *of* $f(z)$ *converges uniformly to* $f(z)$ *in* $\bar{R}'_\gamma$ *except in the neighborhoods of the points* α_k.

If $g(z)$ *is meromorphic on the closed set* $\bar{R}''_\gamma$, *and if all the poles of* $g(z)$ *in this closed set occur in the sequence* β_k, *then the formal development* (24) *of* $g(z)$ *converges uniformly to* $g(z)$ *in* $\bar{R}''_\gamma$ *except in the neighborhoods of the points* β_k

The proof is immediate, by Theorem 7 and the method used in Theorem 5; it is necessary in the proof merely to avoid loci C_γ and $C_{\gamma'}$ on which (25) does not hold uniformly, or on which points β_k or α_k lie

It is desirable in the study of (20) and (24) not to require that (25) should hold uniformly on any closed point set interior to R, for it is advantageous, as we have seen in §8.2, to allow points α_k and β_k to lie interior to R. If such a point does lie interior to R, equation (25) must fail at that point. No limit point of the α_k or β_k can lie interior to R. If P is an arbitrary point of R not on a curve C_γ which passes through a point α_k or β_k, equation (25) is valid uniformly on the boundary of some region bounded by curves C_γ containing in its interior P but no point α_k or β_k, so (25) is valid at P and uniformly valid on the curve C_γ passing through P.

Let $\lambda < \gamma_1$ be the largest number (if existent) such that $f(z)$ is meromorphic interior to R'_λ and all the poles of $f(z)$ interior to R'_λ occur in the sequence α_k. Then the development (20) converges uniformly to $f(z)$ on any closed set inte-

rior to R'_λ containing no point α_k. From §3.4, Theorem 5 we have $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = A \leqq 1/\lambda$. But $A < 1/\lambda$ leads to a contradiction and hence is impossible; choose $\lambda' < \gamma_1$, $A < 1/\lambda' < 1/\lambda$, with no α_k or β_k on $C_{\lambda'}$; we can apply the transformation indicated by (11); if m is suitably chosen each term in the right-hand member of (11) is analytic in $\bar{R}'_{\lambda'}$ and the series converges uniformly in $\bar{R}'_{\lambda'}$ with $\lambda' > \lambda$; this contradicts our hypothesis on $f(z)$. Thus we have $A = 1/\lambda$, and hence the series (20) diverges at every point common to R and R''_λ at which (25) holds. If $f(z)$ is meromorphic interior to every R'_λ, $\lambda < \gamma_1$, and if all the poles of $f(z)$ interior to every such R'_λ occur in the sequence α_k, we merely conclude $A \leqq 1/\gamma_1$.

Similarly, let $\lambda > \gamma_2$ be the smallest number (if existent) such that $g(z)$ is meromorphic interior to R''_λ and all the poles of $g(z)$ interior to R''_λ occur in the sequence β_k. We have $\overline{\lim}_{n\to\infty} |b_n|^{1/n} = B = \lambda$, and the series (24) diverges at every point common to R and R'_λ at which (25) holds. If $g(z)$ is meromorphic interior to every R''_λ, $\lambda > \gamma_2$, and if all the poles of $g(z)$ interior to every such R''_λ occur in the sequence β_k, we merely conclude $B \leqq \gamma_2$.

Condition (25) is sometimes inconvenient in form, for instance due to the fact that infinity is a limit point of the α_k. A condition that may replace (25) is that

$$\lim_{n\to\infty} \left| \rho_1\rho_2 \cdots \rho_n \frac{(z-\beta_1)\cdots(z-\beta_n)}{(z-\alpha_1)\cdots(z-\alpha_n)} \right|^{1/n} = |\Phi(z)| \tag{26}$$

should hold uniformly on each of a set of curves C_γ everywhere dense in R, where the ρ_k are the α_k or are other constants. It is clear that no such ρ_k can vanish if (26) is satisfied, and is clear also that (26) can be used instead of (25) in the proof of (23) and of the conclusion of Theorem 8. We formulate the following remark; a similar remark naturally applies to Theorem 5 and its Corollary 1:

COROLLARY. *Theorem 8 is valid if* (25) *is replaced by* (26).

If a point α_k is infinite, the corresponding factor $z - \alpha_k$ is naturally to be omitted in (25), (26), and similar relations, as well as in (20) and (24).

§8.6. Illustrations

We shall now state some applications of Theorems 6, 7, and 8. More general results than those we mention are obtained in every case by the use of an arbitrary linear transformation of the complex variable, and in some cases by modifying the sequences α_k and β_k without changing the asymptotic properties. We leave to the reader the actual formulation of these suggested results.

Ia. *If the function $f(z)$ is meromorphic for $|z| \leqq R$, if the points β_ν approach zero, if the points α_k become infinite, $\alpha_k \neq \beta_\nu$, and if all the poles of $f(z)$ for $|z| \leqq R$ occur in the sequence α_k, then the formal development* (20) *of $f(z)$ found by interpolation to $f(z)$ converges uniformly to $f(z)$ on any closed set which belongs to the set $|z| \leqq R$ and contains no point α_k.*

Ib. *If the function $g(z)$ is meromorphic for $|z| \geqq R$, if the points β_ν approach zero and the points α_k become infinite, $\alpha_k \neq \beta_\nu$, and if all the poles of $g(z)$ for $|z| \geqq R$ occur in the sequence β_ν, then the formal development (24) of $g(z)$ found by interpolation to $g(z)$ converges uniformly to $g(z)$ on any closed set which belongs to the set $|z| \geqq R$ and contains no point β_ν.*

Under the conditions mentioned, we have

$$\lim_{n\to\infty} \left| \alpha_n \frac{z-\beta_n}{z-\alpha_n} \right| = |z|, \qquad \lim_{n\to\infty} \log \left| \alpha_n \frac{z-\beta_n}{z-\alpha_n} \right| = \log |z|,$$

uniformly for z on any closed limited point set not containing the origin. For z on such a point set containing no point α_k or β_k we have

$$\lim_{n\to\infty} \log \left| \alpha_1 \cdots \alpha_n \frac{(z-\beta_1)\cdots(z-\beta_n)}{(z-\alpha_1)\cdots(z-\alpha_n)} \right|^{1/n} = \log |z|,$$

for this last relation corresponds to the sequence of arithmetic means (of the first order) of the preceding relation. If a number α_k is zero, we omit the corresponding factor α_k; if a number β_k is infinite, we omit the corresponding factor $z - \beta_k$. We are able, then, to apply the Corollary to Theorem 8.

Ia *presents an expansion of an arbitrary meromorphic function, that is to say, a function meromorphic at every finite point of the plane. The expansion is valid in every point at which $f(z)$ is defined, and is found by interpolation in the points β_ν all of which may be chosen at the origin if $f(z)$ has no pole there.* Angelescu [1925] considered the special case of Ia and its generalization by linear transformation, for the expansion of an analytic (not meromorphic) function, and without mention of the properties of the series so far as concerns interpolation. Ia in its present form is due to Walsh [1932a]; it was later proved also by R. Lagrange [1935], but under the unnecessary assumption that the series $\sum 1/|\alpha_k|$ converges. Ia is generalized in Corollary 2 to Theorem 13.

IIa. *Let C be a closed limited point set whose complement K is connected and regular, let the points β_k have no limit point exterior to C, let the relation* (notation of §7.2)

$$\lim_{n\to\infty} |(z-\beta_1)(z-\beta_2)\cdots(z-\beta_n)|^{1/n} = \Delta\,|\phi(z)|$$

hold uniformly on any closed limited point set exterior to C containing no point β_k, and let the points α_ν become infinite, $\alpha_\nu \neq \beta_k$. If $f(z)$ is meromorphic on and within C_R, and if all the poles of $f(z)$ on and within C_R occur in the sequence α_k, then the formal development (20) *converges uniformly to $f(z)$ on any closed point set which consists of points on or within C_R and which contains no point α_k.*

IIb. *Let C and the points α_k and β_ν satisfy the hypothesis of* IIa. *If the function $g(z)$ is meromorphic on and exterior to C_R, and if all the poles of $g(z)$ on and exterior to C_R occur in the sequence β_ν, then the formal development* (24) *of $g(z)$ converges uniformly to $g(z)$ on any closed point set which consists of points on and exterior to C_R and which contains no point β_ν.*

IIa and IIb are generalizations of both IIa and IIb, and IIIa and IIIb respectively of §8.4. The proof of the present IIa and IIb is immediate from the Corollary to Theorem 8, by virtue of the following uniform relations on any closed limited point set containing no point α_k:

$$\lim_{n\to\infty} \left| \frac{\alpha_n}{z - \alpha_n} \right| = 1, \qquad \lim_{n\to\infty} \log \left| \frac{\alpha_n}{z - \alpha_n} \right| = 0,$$

$$\lim_{n\to\infty} \log \left| \frac{\alpha_1\alpha_2 \cdots \alpha_n}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)} \right|^{1/n} = 0.$$

If a number α_k is zero, we omit the corresponding factor α_k.

We call attention to the fact that §7.2, Theorem 2, Corollary 3 and the method of §4.4 have important application here in the determination of sequences β_k. Given any set of points interior to K but with no limit point in K, the set β_k can be chosen to contain all of these points. Similar remarks apply frequently in the sequel.

IIIa. *If the function $f(z)$ is meromorphic for $|z| \leqq R$, $B < R < A$, if all the poles of $f(z)$ for $|z| \leqq R$ occur in the sequence α_k, if the points α_k have no limit point interior to $|z| = A$ and satisfy the condition*

$$\lim_{n\to\infty} |(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)|^{1/n} = A$$

uniformly on any closed set interior to $|z| = A$ containing no point α_k, and if the points $\beta_k \neq \alpha_\nu$ have no limit point exterior to $|z| = B$ and satisfy the condition

$$\lim_{n\to\infty} |(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)|^{1/n} = |z|$$

uniformly on any closed set exterior to $|z| = B$ containing no point β_k, then the formal development (20) *of $f(z)$ found by interpolation converges to $f(z)$ uniformly on any closed set which lies in the set $|z| \leqq R$ and contains no point α_k.*

IIIb. *Let the points α_k and β_k satisfy the hypothesis of* IIIa. *If the function $g(z)$ is meromorphic for $|z| \geqq R$, $B < R < A$, and if all the poles of $g(z)$ for $|z| \geqq R$ occur in the sequence β_k, then the formal development* (24) *of $g(z)$ found by interpolation converges to $g(z)$ uniformly on any closed set which lies in the set $|z| \geqq R$ and contains no point β_k.*

The hypothesis of IIIa is satisfied if the points α_k and β_k are uniformly distributed on $|z| = A$ and $|z| = B$ respectively with respect to arc length in the sense of §7.5; see §7.6.

IVa. *If the function $f(z)$ is analytic for $1/R \leqq |z| \leqq R > 1$, then the sequence of rational functions of respective degrees $2n$ whose poles lie in the origin and the point at infinity each of multiplicity n and which interpolate to $f(z)$ in points $\beta_1, \beta_2, \cdots, \beta_{2n+1}$, where the points β_k are uniformly distributed on $|z| = 1$ with respect to arc length as parameter, converges to $f(z)$ uniformly for $1/R \leqq |z| \leqq R$.*

IVb. *If the function $g(z)$ is analytic for $|z| \leqq 1/R$ and for $|z| \geqq R > 1$, then the sequence of rational functions of respective degrees $2n - 1$ whose poles lie in points $\beta_1, \beta_2, \cdots, \beta_{2n-1}$, where the β_k are uniformly distributed on $|z| = 1$, and which interpolate to $g(z)$ in the origin and at the point at infinity each of multiplicity n, converges to $g(z)$ uniformly for $|z| \leqq 1/R$ and $|z| \geqq R$.*

The hypothesis on the points β_k is satisfied provided $\beta_k = \beta^k$, $|\beta| = 1$, where β is no root of unity (compare §7.6). For this case, the expansion in IVa reduces on $|z| = 1$ to a trigonometric series which is real when $f(z)$ is real, an expansion due to Euler and used by Leverrier in applications to astronomy; the proposition is essentially due to Pólya [1934], established in answer to a question formulated by Wintner.

IVa and IVb as stated refer to a series of interpolation with the terms grouped in pairs. The corresponding results hold without this grouping.

§8.7. Harmonic functions as generating functions

The illustrations that we have presented in §§8.4 and 8.6 are only the most obvious ones; the choice is more or less accidental, depending on the convenience of certain elementary functions. We shall now set forth a general method [Walsh, 1934c] for obtaining sequences α_{nk} and β_{nk} which satisfy such conditions as (16) and (17). Interesting applications are far too numerous even to be stated in detail; we shall therefore emphasize methods rather than results. Implications for duality are frequent and obvious.

THEOREM 9. *Let C_1 and C_2 be two non-intersecting analytic Jordan curves or non-intersecting sets each of a finite number of non-intersecting analytic Jordan curves; suppose a region R is bounded by the whole of C_1 and C_2, and suppose for definiteness that one curve of the set C_1 contains all the other curves of the set C_1 and all the curves of C_2. Then there exist point sets α'_{nk} and β'_{nk} on C_1 and C_2 respectively such that uniformly on every closed set interior to R equations (16) are valid with*

$$|\Phi(z)| \equiv \left|\frac{\Phi_2(z)}{\Phi_1(z)}\right| \equiv \exp\left[\frac{-2\pi}{\tau}\,U(x, y)\right], \quad \tau \text{ constant}, \tag{27}$$

where $U(x, y)$ is harmonic in R, continuous in the corresponding closed region, and takes on the values zero and unity on C_1 and C_2 respectively.

Consequently, if $f(z)$ is analytic in the closed region or set of regions R_μ bounded by the complete locus $U(x, y) = \mu$, $0 < \mu < 1$, in R, where R_μ contains in its interior no point of C_1, then the sequence of rational functions $r_n(z)$ of respective degrees n whose poles lie in the points $\alpha_{nk} = \alpha'_{nk}$ which is found by interpolation to $f(z)$ in the points $\beta_{nk} = \beta'_{n+1,k}$ converges to $f(z)$ uniformly in R_μ. If $f(z)$ is analytic merely interior to such a region or set of regions R_μ, then the sequence $r_n(z)$ converges to $f(z)$ uniformly on any closed point set interior to R_μ.

Our method is essentially a generalization of the methods of Hilbert (§§4.2 and 4.5) and Fejér (§§4.3 and 7.6). We start with the formula

$$(28)\quad U(x, y) = \frac{1}{2\pi}\int_{C_1}\left(U\,\frac{\partial \log r}{\partial\nu} - \log r\,\frac{\partial U}{\partial\nu}\right) ds$$
$$+ \frac{1}{2\pi}\int_{C_2}\left(U\,\frac{\partial \log r}{\partial\nu} - \log r\,\frac{\partial U}{\partial\nu}\right) ds, \qquad r^2 = (x - x')^2 + (y - y')^2;$$

the running coordinates are (x', y'), ν indicates the exterior normal for R, and integration is taken over C_1 and C_2 in the positive sense with respect to the region R. Equation (28) is then valid for every (x, y) interior to R. But $U(x', y')$ vanishes on C_1, and we have

$$\int_{C_2} U\,\frac{\partial \log r}{\partial\nu}\, ds = \int_{C_2}\frac{\partial \log r}{\partial\nu}\, ds = 0, \qquad (x, y) \text{ in } R,$$

since for (x, y) in R the function $\log r$ is harmonic on and within each of the constituent curves of C_2. Hence (28) can be written in the form

$$(29)\quad U(x, y) = -\frac{1}{2\pi}\int_{C_1}\log r\,\frac{\partial U}{\partial\nu}\, ds - \frac{1}{2\pi}\int_{C_2}\log r\,\frac{\partial U}{\partial\nu}\, ds, \qquad (x, y) \text{ in } R.$$

We introduce a new variable σ by the relations

$$d\sigma = -(\partial U/\partial\nu)ds, \text{ on } C_1; \qquad d\sigma = (\partial U/\partial\nu)ds, \text{ on } C_2;$$

this is allowable, for the function $U(x, y)$ can be extended harmonically from R across C_1 and C_2. No point of C_1 or C_2 can be a critical point of $U(x, y)$, for at a critical point the tangents to the level curves of a harmonic function are equally spaced, and no such level curves can enter the region R. At every point of C_1 and C_2 we have $\partial U/\partial s = 0$; interior to R we have $0 < U < 1$, so we must have $\partial U/\partial\nu < 0$ on C_1, $\partial U/\partial\nu > 0$ on C_2. We introduce the notation

$$\int_{C_1} d\sigma = \tau = \int_{C_2} d\sigma, \qquad \tau > 0,$$

and we may set $0 \leqq \sigma \leqq \tau$ on C_1, $\tau < \sigma \leqq 2\tau$ on C_2.

Equation (29) can now be written in the form (compare §§4.2 and 4.3)

$$(30)\quad U(x, y) = \frac{1}{2\pi}\int_0^{\tau}\log r\, d\sigma - \frac{1}{2\pi}\int_{\tau}^{2\tau}\log r\, d\sigma$$
$$= \frac{\tau}{2\pi}\lim_{n\to\infty}\frac{\log r_{n1} + \log r_{n2} + \cdots + \log r_{nn}}{n}$$
$$-\frac{\tau}{2\pi}\lim_{n\to\infty}\frac{\log \rho_{n1} + \log \rho_{n2} + \cdots + \log \rho_{nn}}{n},$$

where $r_{nk} = | z - \alpha'_{nk} |$ corresponds to the point α'_{nk}: $\sigma = k\tau/n$ on C_1 and $\rho_{nk} = | z - \beta'_{nk} |$ corresponds to the point β'_{nk}: $\sigma = (k + n)\tau/n$ on C_2. The convergence in (30) is uniform for z on any closed set interior to R, and (30) is equivalent to (16) and (27). The remainder of Theorem 9 follows from Theorem 5, Corollary 1.

The analogue of Corollary 2 to Theorem 5 is valid; we have for $\mu' > \mu$

$$\overline{\lim_{n\to\infty}} [\max | f(z) - r_n(z) | , z \text{ on } R_{\mu'}]^{1/n} \leqq e^{-2\pi(\mu'-\mu)/\tau}.$$

Equation (30) holds with $r_{nk} = | z - \alpha'_{nk} |$ and $\rho_{nk} = | z - \beta'_{nk} |$, as do equations (16) and (27), whenever the points α'_{nk} and β'_{nk} are uniformly distributed on C_1 and C_2 respectively with respect to the parameter σ. In particular the α'_{nk} and β'_{nk} may be chosen not to depend on n; the expansion of $f(z)$ is a series of interpolation.

Theorem 9 is of especial interest in case $f(z)$ has C_1 or a locus $U = \mu$, $0 < \mu < 1$, as a natural boundary, for the expansion found by interpolation is then valid throughout the domain of definition of the function.

In the geometric situation of Theorem 9, suppose we take as point of departure not the function $U(x, y)$ but a function $U_1(x, y)$ harmonic interior to R, continuous in the corresponding closed region, assuming values γ_1 and $\gamma_2 \neq \gamma_1$ on C_1 and C_2 respectively, the values γ_1 and γ_2 being constant but not necessarily zero and unity. Of course we have $U(x, y) \equiv [U_1(x, y) - \gamma_1]/(\gamma_2 - \gamma_1)$. When $U(x, y)$ is replaced by $U_1(x, y)$, equation (28) remains valid, but equation (29) is to be modified by adding the term

$$\frac{1}{2\pi} \int_{C_1} U_1(x, y) \frac{\partial \log r}{\partial \nu} ds = \gamma_1$$

to the right-hand member. The points α'_{nk} and β'_{nk} uniformly distributed on C_1 and C_2 respectively with respect to the parameter σ are also uniformly distributed on C_1 and C_2 with respect to the parameter σ_1, $d\sigma_1 = \pm (\partial U_1/\partial \nu)ds$, for the parameters σ and σ_1 are linear functions of each other. Equations (16) and (27) are valid in their original forms. Of course (27) can readily be expressed in terms of $U_1(x, y)$ instead of $U(x, y)$.

We have by no means exhausted the possibilities of the method used in Theorem 9. Let us prove, for instance, the

COROLLARY. *Suppose in addition to the region R of Theorem 9 we have a region R' bounded by C_1' and C_2' satisfying analogous conditions, with analogous notation, where C_1 and C_1' are mutually exterior. Let $U'(x, y)$ denote the new analogue of $U(x, y)$. Then there exist points α'_{nk} on C_1 and C_1' and points β'_{nk} on C_2 and C_2' respectively such that* (16) *is valid simultaneously interior to both R and R'; we have* (27) *and its natural analogue* (27'). *Consequently, if $f(z)$ is analytic in the closed regions or sets of regions R_μ and R'_μ, then the sequence of rational functions of respective degrees n whose poles lie in the points $\alpha_{nk} = \alpha'_{nk}$ found by interpolation to $f(z)$ in the points $\beta_{nk} = \beta'_{n+1,k}$ converges to $f(z)$ uniformly in R_μ and R'_μ.*

The right-hand member of equation (28) represents a function identically zero for (x, y) exterior to R, for under such conditions the functions $U(x', y')$ and $\log r$ are both harmonic throughout R. For (x, y) in R', the right-hand member of (29) similarly represents a function identically zero. Hence the expression

$$-\frac{1}{2\pi}\int_{C_1}\log r\,\frac{\partial U}{\partial \nu}\,ds-\frac{1}{2\pi}\int_{C_2}\log r\,\frac{\partial U}{\partial \nu}\,ds$$
$$-\frac{1}{2\pi}\int_{C_1'}\log r\,\frac{\partial U'}{\partial \nu}\,ds-\frac{1}{2\pi}\int_{C_2'}\log r\,\frac{\partial U'}{\partial \nu}\,ds$$

represents the function $U(x, y)$ for (x, y) in R and the function $U'(x, y)$ for (x, y) in R'. The proof now goes through as before, if we set $d\sigma$ equal to $-(\partial U/\partial \nu)ds$, $(\partial U/\partial \nu)ds$, $-(\partial U'/\partial \nu)ds$, $(\partial U'/\partial \nu)ds$ on C_1, C_2, C_1', C_2', respectively,

$$\tau=\int_{C_1}d\sigma+\int_{C_1'}d\sigma=\int_{C_2}d\sigma+\int_{C_2'}d\sigma\,.$$

The Corollary clearly extends to the case where the two regions R and R' are replaced by any finite number of such mutually exterior regions.

Even if we do not assume that C_1 and C_1' are mutually exterior, the points α_{nk}' and β_{nk}' exist so that (16), (27), and (27′) are valid; but the topological properties demanded in Corollary 1 to Theorem 5 are not exhibited. However, other properties are possessed by this configuration, as we prove in Theorem 10.

§8.8. Harmonic functions as generating functions, continued

Still another application, similar to the Corollary to Theorem 9, of the method of §8.7 is of interest; we choose a restricted situation for simplicity.

THEOREM 10. *Let C_1, C_2, C_3, C_4 be four analytic Jordan curves, with C_4 interior to C_3, no point of C_3 exterior to C_2, and C_2 interior to C_1. Let R and R' denote the annular regions bounded respectively by C_1 and C_2, and by C_3 and C_4. Let $U(x, y)$ denote the function harmonic interior to R, continuous in the corresponding closed region, and taking on the values zero and unity on C_2 and C_1 respectively. Let $U'(x, y)$ denote the function harmonic interior to R', continuous in the corresponding closed region, and taking on the values zero and unity on C_3 and C_4 respectively. Then there exist points α_{nk}' on C_2 and C_3 and points β_{nk}' on C_1 and C_4 such that we have equations* (16) *valid uniformly on any closed set interior to R or R', with*

$$|\Phi(z)|\equiv\left|\frac{\Phi_2(z)}{\Phi_1(z)}\right|\approx\nu\exp\left[\frac{-2\pi}{\tau}U(x,y)\right]\qquad in\ R\,,\tag{31}$$
$$|\Phi(z)|\approx\nu\exp\left[\frac{-2\pi}{\tau}U'(x,y)\right]\qquad in\ R'\,,$$

where ν and τ are positive constants.

Consequently if $f(z)$ is analytic in the two closed regions R'_μ bounded by and exterior to the curve $U(x, y) = \mu$, $0 < \mu < 1$, and bounded by and interior to the curve $U'(x, y) = \mu$ respectively, then the sequence of rational functions of respective degrees n whose poles lie in the points $\alpha_{nk} = \alpha'_{nk}$ and which is found by interpolation to $f(z)$ in the points $\beta_{nk} = \beta'_{n+1,k}$ converges to $f(z)$ uniformly in R'_μ. If $g(z)$ is analytic in the closed annular region R_μ bounded by the two curves $U(x, y) = \mu$, $0 < \mu < 1$, and $U'(x, y) = \mu$, then the sequence of rational functions of respective degrees n whose poles lie in the points $\beta_{nk} = \beta'_{nk}$ which is found by interpolation to $g(z)$ in the points $\alpha_{nk} = \alpha'_{n+1,k}$ converges to $g(z)$ uniformly in R_μ.

From (28) we find as before

$$(32)\quad -\frac{1}{2\pi}\int_{C_3} \log r \frac{\partial U'}{\partial \nu}\, ds - \frac{1}{2\pi}\int_{C_4} \log r \frac{\partial U'}{\partial \nu}\, ds = \begin{cases} U'(x, y), & (x, y) \text{ in } R', \\ 0, & (x, y) \text{ in } R. \end{cases}$$

We have, however,

$$\frac{1}{2\pi}\int_{C_1} U \frac{\partial \log r}{\partial \nu}\, ds = \frac{1}{2\pi}\int_{C_1} \frac{\partial \log r}{\partial \nu}\, ds = 1, \qquad (x, y) \text{ interior to } C_1,$$

whence

$$(33)\qquad -\frac{1}{2\pi}\int_{C_1} \log r \frac{\partial U}{\partial \nu}\, ds - \frac{1}{2\pi}\int_{C_2} \log r \frac{\partial U}{\partial \nu}\, ds$$
$$= \begin{cases} U(x, y) - 1, & (x, y) \text{ in } R, \\ -1, & (x, y) \text{ in } R'. \end{cases}$$

Let us introduce a new variable σ by setting $d\sigma$ equal to $(\partial U/\partial \nu) ds$, $-(\partial U/\partial \nu) ds$, $-(\partial U'/\partial \nu) ds$, $(\partial U'/\partial \nu) ds$ respectively on C_1, C_2, C_3, C_4; then we have

$$\tau = \int_{C_1} d\sigma + \int_{C_4} d\sigma = \int_{C_2} d\sigma + \int_{C_3} d\sigma;$$

it will be noticed that each of these four differentials and integrals is positive. We may set $0 \leqq \sigma < \tau$ on C_2 and C_3, $\tau < \sigma \leqq 2\tau$ on C_1 and C_4. We can evaluate the sum of the left-hand members of (32) and (33), as in (30), and obtain

$$(34)\quad \frac{\tau}{2\pi} \lim_{n\to\infty} \frac{\log r_{n1} + \log r_{n2} + \cdots + \log r_{nn}}{n}$$
$$- \frac{\tau}{2\pi} \lim_{n\to\infty} \frac{\log \rho_{n1} + \log \rho_{n2} + \cdots + \log \rho_{nn}}{n} = \begin{cases} U(x, y) - 1, & (x, y) \text{ in } R, \\ U'(x, y) - 1, & (x, y) \text{ in } R', \end{cases}$$

where $r_{nk} = |z - \alpha'_{nk}|$ corresponds to the point α'_{nk}: $\sigma = k\tau/n$ on C_2 or C_3 and $\rho_{nk} = |z - \beta'_{nk}|$ corresponds to the point β'_{nk}: $\sigma = (k + n)\tau/n$ on C_1 or C_4.

The convergence in (34) is uniform on any closed set interior to R or R', and of course (34) can be written in the form (16) with (31) true, where $v = e^{2\pi/\tau}$.

Equations (34), (16), and (31) hold whenever the points α'_{nk} are uniformly distributed on C_2 and C_3 with respect to the parameter σ and the points β'_{nk} are uniformly distributed on C_1 and C_4 with respect to σ.

Limiting cases of Theorem 10, in which the curves C_2 and C_3 coincide and C_1 and C_4 shrink to points, are given as §8.4, Va and Vb, and §8.6, IVa and IVb.

It will be noted that in Theorem 10 the two curves C_2 and C_3 may coincide; in this case the points α'_{nk} may simply be chosen uniformly distributed on C_2 with respect to the parameter σ', $d\sigma' = -[(\partial U/\partial \nu) + (\partial U'/\partial \nu)]ds$. We have then an expansion in the neighborhood of an arbitrary analytic Jordan curve C_2 of an arbitrary function analytic on C_2; the curves C_1 and C_4 may be chosen in a variety of ways, and the function $f(z)$ enters into the expansion merely by means of its values on C_2. Similarly, if we start with an arbitrary annulus bounded by two analytic Jordan curves C_1 and C_4, an arbitrary function $f(z)$ analytic in that annulus can be expanded throughout the entire annulus, and the function $f(z)$ is involved in the expansion merely by its values on an arbitrary analytic Jordan curve $C_2 = C_3$ which separates C_1 and C_4.

Theorem 10 extends at once to the case that C_1 and C_4 are sets of curves instead of single curves. If the region R is bounded by several Jordan curves, either a single region or several regions may simultaneously play the rôle of the region R' of Theorem 10.

Let us present one further detailed illustration to show how sequences of interpolation can be obtained by starting with harmonic functions as point of departure; the geometric situation is a limiting case of that of Theorem 10.

Let C be an analytic Jordan curve of the z-plane containing the origin in its interior, let $w = \phi(z)$ map the exterior of C onto $|w| > 1$ so that the points at infinity correspond to each other, and let $w = \chi(z)$ map the interior of C onto $|w| > 1$ so that $z = 0$ corresponds to $w = \infty$. We have (§4.3) uniformly for z on any closed set exterior to C

$$\lim_{n\to\infty} |(z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{nn})|^{1/n} = \Delta\,|\phi(z)|, \qquad \Delta = 1/|\phi'(\infty)|,$$

where the points β_{nk} are uniformly distributed on C with respect to σ, $d\sigma = (2\pi)^{-1}(\partial G/\partial \nu)ds$, where $G(x, y)$ is Green's function for the exterior of C with pole at infinity, ν is the exterior normal of C, and where Δ is the capacity of C. On any closed set interior to C we have (§4.3)

$$\lim_{n\to\infty} |(z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{nn})|^{1/n} = \Delta\,.$$

Similarly, let points β'_{nk} be chosen on C uniformly distributed with respect to σ', $d\sigma' = -(2\pi)^{-1}(\partial G'/\partial \nu)\,ds$, where ν is the exterior normal, and where $G'(x, y)$ is Green's function for the interior of C with pole at the origin O. That is to say, $G'(x, y)$ is harmonic interior to C except at O, is continuous in the corresponding closed region except at O, vanishes on C, and in the neighbor-

hood of O can be expressed as $-\frac{1}{2}\log(x^2+y^2)$ plus a function harmonic in that neighborhood. A representation for $G'(x, y)$ interior to C is found from (28):

$$G'(x,y) = \frac{-1}{2\pi}\int_C \log r \frac{\partial G'}{\partial \nu}\, ds - \log r_0, \qquad r_0 = (x^2+y^2)^{1/2};$$

this right-hand member represents the function zero exterior to C. Then we have uniformly on any closed set interior to C not containing the origin

$$\lim_{n\to\infty}\left|\frac{(z-\beta'_{n1})(z-\beta'_{n2})\cdots(z-\beta'_{nn})}{z^n}\right|^{1/n} = |\chi(z)| = e^{G'},$$

and uniformly on any closed limited set exterior to C we have

$$\lim_{n\to\infty}\left|\frac{(z-\beta'_{n1})(z-\beta'_{n2})\cdots(z-\beta'_{nn})}{z^n}\right|^{1/n} = 1.$$

Let us now set

$$\beta''_{2n,k} = \beta_{nk} \text{ when } k \leqq n, \qquad \beta''_{2n,n+k} = \beta'_{nk} \text{ when } k \leqq n;$$

it is sufficient also to require that the $\beta''_{2n,k}$ be uniformly distributed on C with respect to the parameter σ'', $d\sigma'' = d\sigma + d\sigma'$. We find from the previously derived relations

$$\lim_{n\to\infty}\left|\frac{(z-\beta''_{2n,1})\cdots(z-\beta''_{2n,2n})}{z^n}\right|^{1/(2n)} = \begin{cases}\Delta^{1/2}\,|\phi(z)|^{1/2}, & z \text{ exterior to } C,\\ \Delta^{1/2}\,|\chi(z)|^{1/2}, & z \text{ interior to } C;\end{cases}$$

these limits hold uniformly on any closed limited set exterior to C and on any closed set interior to C not containing the origin respectively. Indeed, we need merely to take logarithms and to note that the relations $\lim_{n\to\infty}\mu_n = \mu$, $\lim_{n\to\infty}\mu'_n = \mu'$ imply $\lim_{n\to\infty}[(\mu_n+\mu'_n)/2] = (\mu+\mu')/2$.

The process just indicated can be described as that of "mixing" the points β_{nk} and β'_{nk} in *equal* proportions; in terms of parameters, it means setting $d\sigma'' = d\sigma + d\sigma'$ instead of using various multiples of $d\sigma$ and $d\sigma'$. It is likewise possible to use other proportions of β_{nk} and β'_{nk}. For instance, let λ be arbitrary, $0 < \lambda < 1$. There exist monotonically non-decreasing sequences of positive integers n_j and n'_j becoming infinite and such that

$$\lim_{j\to\infty} n_j/(n_j+n'_j) = \lambda.$$

As an example we can choose the sequences so that $n_j + n'_j$ takes on all the values 1, 2, $\cdots$; it is sufficient to set $n_j + n'_j = n$, $n_j = [\lambda n]$, that is, n_j is the largest integer not greater than λn. We have

$$\lambda - \frac{n_j}{n} = \frac{\lambda n - [\lambda n]}{n},$$

which is not less than zero and not greater than $1/n$, hence approaches zero We do not in the sequel suppose that $n_j + n'_j$ necessarily takes on all values.

The two general relations $\lim \mu_n = \mu$, $\lim \mu'_n = \mu'$ now imply

$$\lim(n_j \mu_{n_j} + n'_j \mu'_{n'_j})/(n_j + n'_j) = \lambda\mu + (1 - \lambda)\mu',$$

so we may set

$$\beta''_{nk} = \beta_{n_j, k}, \text{ for } k \leqq n_j, \qquad \beta''_{n, n_j+k} = \beta'_{n'_j, k}, \text{ for } k \leqq n'_j;$$

if we prefer we may require directly that the points β''_{nk} be uniformly distributed on C with respect to the parameter σ'', $d\sigma'' = \lambda\, d\sigma + (1 - \lambda)d\sigma'$, adding the equations for $G(x, y)$ and $G'(x, y)$ after multiplying by λ and $1 - \lambda$ respectively. Either requirement yields

$$(35) \qquad \lim_{n\to\infty} \left| \frac{(z - \beta''_{n1}) \cdots (z - \beta''_{nn})}{z^{n'_j}} \right|^{1/n}$$

$$= | \Phi(z) | \equiv \begin{cases} \Delta^\lambda \, | \phi(z) |^\lambda, & z \text{ exterior to } C, \\ \Delta^\lambda \, | \chi(z) |^{1-\lambda}, & z \text{ interior to } C, \end{cases}$$

uniformly on any closed limited set exterior to C and on any closed set interior to C not containing the origin.

Corollary 1 to Theorem 5 is applicable. The locus $| \Phi(z) | = R$ consists here of the curves $| \phi(z) | = \Delta^{-1} R^{1/\lambda}$ and $| \chi(z) | = \Delta^{-\lambda/(1-\lambda)} R^{1/(1-\lambda)}$. A function analytic in the regions $| \phi(z) | > \Delta^{-1} R^{1/\lambda}$ and $| \chi(z) | > \Delta^{-\lambda/(1-\lambda)} R^{1/(1-\lambda)}$ can be expanded by interpolation in the points 0 and ∞ of respective multiplicities n'_j and $n_j + 1$, by rational functions of respective degrees $n = n_j + n'_j$ with poles in the points β''_{nk}. A function analytic in the region $| \phi(z) | < \Delta^{-1} R^{1/\lambda}$ and $| \chi(z) | < \Delta^{-\lambda/(1-\lambda)} R^{1/(1-\lambda)}$ (with the curve C included) can be expanded by interpolation in the points β''_{nk}, with rational functions of respective degrees $n = n_j + n'_j$ with poles at infinity of order $n_j - 1$ and in the origin of order n'_j.

In the case that C is a circle, the points β_{nk} and also the points β'_{nk} may be chosen as the n-th roots of unity or as the n-th roots of any constant or variable number A_n of modulus unity. Indeed, it may be verified directly from the asymptotic formulas that the β''_{nk} themselves may be chosen as the n-th roots of any constant or variable number of modulus unity. If in particular we choose $\lambda = 1/2$ and the β''_{nk} roots of unity, we have §8.4, Theorems Va and Vb.

Our discussion, here suggested to apply to a single Jordan curve, can also be applied without essential change simultaneously to several mutually exterior Jordan curves.

We have presented in §§8.7 and 8.8 merely a few illustrations of a method. Further illustrations, analogous to some of the preceding results of the present and of other chapters, will readily occur to the reader. For instance, further results can be obtained (i) by using the methods of the Corollary of Theorem 9, of Theorem 10, and of (35) to combine (in various proportions) simpler configurations already studied; (ii) by removing the restriction on the boundaries of the

regions; throughout §§8.7 and 8.8 the given Jordan curves need not be taken as analytic, provided the integrals are taken in the sense of §§7.6 and 9.12; those Jordan curves may frequently be replaced by Jordan arcs; the method used in §4.4 enables us to treat any region with certain topological properties provided merely the harmonic function $U(x, y)$ exists; the regions may be finite or infinite; (iii) by formulating the duals of these various results; for instance the dual of the Corollary to Theorem 9 leads to an expansion valid in a multiply connected region; (iv) by elaborating the remark at the end of §8.4 relative to the components of a given function, and applying it in these situations; (v) by studying series of interpolation and allowing the functions expanded to be meromorphic instead of analytic; (vi) by allowing some of the points α_k or β_k or both in the series of interpolation to lie *interior* to the regions in which functions are given, so that the given functions may be meromorphic in those regions and still expansible; (vii) by taking as point of departure not a region and a function depending on it as in Theorem 9, but a harmonic function directly, as in §§8.4 and 8.6 and the derivation of (35); such a harmonic function may become infinite in one or more points without seriously altering the application. All of these suggested results are now more or less immediate, thanks to the methods already developed, and are left to the reader.

The significance of the results just suggested for the general problem of the representation of functions is in part as follows. We may start with an arbitrary finite region R, say bounded by a finite number of non-intersecting Jordan curves. An arbitrary function $f(z)$ analytic in R can be represented in R uniformly on any closed set interior to R by a sequence of rational functions $r_n(z)$ found by interpolation to $f(z)$. All the points of interpolation may be chosen identical, in which case the sequence $r_n(z)$ involves the derivatives of $f(z)$ at a point, or those points of interpolation may be chosen (for each n) all distinct, in which case $r_n(z)$ involves functional values of $f(z)$ but no derivatives. The points of interpolation and the prescribed poles of the $r_n(z)$ depend on R but not on $f(z)$; the points of interpolation may be chosen on an arbitrary Jordan arc or curve C in R (which may even separate components of the boundary of R), so the sequence $r_n(z)$ depends on $f(z)$ merely through the values of $f(z)$ on C. If a function $f(z)$ is analytic on C but not throughout R, the sequence $r_n(z)$ found by interpolation still exists and converges uniformly to $f(z)$ in the neighborhood of C, in fact uniformly on any closed set interior to a definite region interior to R and containing C which depends only on R, C, and the singularities of $f(z)$. If poles α_{nk} and points of interpolation β_{nl} for $r_n(z)$ are chosen independent of n, as may always be done, the sequence $r_n(z)$ is found as the partial sums of a series of interpolation (20). If $f(z)$ is analytic on C but not throughout R, the regions of convergence and of divergence in R of the sequence $r_n(z)$ are then separated by a specific locus which depends only on R, C, and the singularities of $f(z)$, and such regions can be uniquely defined from R, C, and $\overline{\lim}_{n\to\infty} | a_n |^{1/n}$ in (20).

The study of asymptotic conditions of form (16), with significance for the problem of interpolation, is to be resumed in Chapter IX.

§8.9. Geometric conditions on given points

We have already met various conditions for convergence of sequences and series of interpolation, conditions expressed in such a general form as (12) or in terms of (16). We turn now to the discussion of conditions [Walsh, 1932, 1932a, 1934b] which are expressed directly in geometric language on the points α_{nk} and β_{nk}.

THEOREM 11. *Let R_1, R_2, R_3 be closed point sets, and let R be the locus of all points t when α, β, z have R_1, R_2, R_3 as their respective loci and the relation*

$$(36) \qquad |(t, \alpha, z, \beta)| \geqq 1, \qquad (t, \alpha, z, \beta) \equiv \frac{(t-\alpha)(z-\beta)}{(z-\alpha)(t-\beta)},$$

holds. Let the points α_{nk} have no limit point exterior to R_1, the points β_{nk} no limit point exterior to R_2. If the function $f(z)$ is analytic in R, then the sequence of rational functions $r_n(z)$ of form (8) which interpolate to $f(z)$ in the points β_{nk} converges to $f(z)$ uniformly in R_3.

The point set R may be the whole plane. This case is trivial, for $f(z)$ must be a constant, and every $r_n(z)$ is identically equal to $f(z)$. This case is henceforth excluded.

Whenever the cross-ratio (t, α, z, β) is indeterminate, or whenever the denominator in the equation of (36) vanishes, we consider (36) satisfied; this is a natural convention, as one may find by studying the relationships involved. The set R is the whole plane, under this convention, whenever R_1 and R_2 have a common point.

The point set R is closed; this follows from the closure of the sets R_1, R_2, R_3 and the continuity of the function (t, α, z, β). It may be similarly proved that R is a closed region if R_1, R_2, and R_3 are closed regions.

The locus R contains the set R_2, for when $t = \beta$ the convention already made includes the validity of (36). The locus R contains the set R_3, for when $t = z$ the cross-ratio (t, α, z, β) if defined takes the value unity.

The locus R depends continuously on the point sets R_1, R_2, R_3, in the sense that a slight enlargement of R_1, R_2, R_3 causes only a slight enlargement of R, for (36) is a continuous relation which defines t (the trivial case that z and α may be equal has been excluded, because then R is the entire plane), uniformly continuous on suitably chosen closed point sets containing R_1, R_2, R_3 in their respective interiors. The given function $f(z)$ is analytic in R, hence analytic in some closed set R' containing R in its interior; we choose R' as a finite number of mutually exclusive closed regions each bounded by a finite number of non-intersecting contours Γ. There exist closed sets R_1', R_2', R_3' which contain the sets R_1, R_2, R_3 in their respective interiors, and such that the locus of all points t satisfying (36) when α, β, z have R_1', R_2', R_3' as their respective loci lies interior to R'. The sets R_2' and R_3' belong to the locus of t, hence must lie interior to R'. Then for α, β, z in R_1', R_2', R_3' respectively and for t on Γ we have uniformly

$$(37) \qquad |(t, \alpha, z, \beta)| \leqq p < 1,$$

where p is suitably chosen. The uniformity follows from the closure of the point sets involved.

In the proof of Theorem 11 we assume, as we may do, that R' is finite. For suitably large n, the points α_{nk} and β_{nk} lie in R_1' and R_2' respectively; by (37) we have $\alpha_{nj} \neq \beta_{nk}$. Equation (4) is valid for z on R_3', since both R_2' and R_3' lie interior to R'. For z in R_3', $\beta_{n,n+1}$ in R_2', and t on Γ, the expressions

$$|(z - \beta_{n,n+1})/(t - \beta_{n,n+1})|$$

are uniformly limited for all n, say less than μ. Then the expression which appears in (12) is in absolute value less than $p^n\mu$ for t on Γ and for z on R_3'. Theorem 11 follows at once from Theorem 3; indeed we have proved uniform convergence of $r_n(z)$ to $f(z)$ not merely in R_3 but also in a region R_3' which contains R_3 in its interior.

It is worth noting that under the geometric conditions of Theorem 11, failure of analyticity of $f(z)$ at a boundary point of R may cause failure of convergence to $f(z)$ of the formal development of $f(z)$, at a point z' of R_3. We need merely choose $\alpha, \beta, z' \neq \beta, t$ (they exist in their proper loci): $f(z) = 1/(t - z)$, $\alpha_{nj} = \alpha$, $\beta_{nk} = \beta$, where $|(t, \alpha, z', \beta)| = 1$. We have

$$f(z) - r_n(z) = \frac{(z - \beta)^{n+1}(t - \alpha)^n}{(t - \beta)^{n+1}(z - \alpha)^n(t - z)},$$

and for $z = z'$ the right-hand member fails to approach zero as a limit. An obvious modification is to be made in this example if t is infinite.

The method of proof of Theorem 11 yields

COROLLARY 1. *If, under the hypothesis of Theorem 11, the function $f(z)$ is analytic in the point set which is the locus of all points t where α, β, z have R_1, R_2, R_3 as their respective loci and the relation $|(t, \alpha, z, \beta)| \geqq p < 1$ obtains, then we have*

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - r_n(z)|,\ z \text{ on } R_3]^{1/n} \leqq p.$$

Certain special cases of Theorem 11 are worth stating explicitly

COROLLARY 2. *Let the function $f(z)$ be analytic for $|z| < T$, let the numbers α_{nk} have no limit point whose modulus is less than A, and the numbers β_{nk} have no limit point whose modulus is greater than $B < T$. Then the sequence $r_n(z)$ converges to $f(z)$ for $|z| < (AT - BT - 2AB)/(A - B + 2T)$, uniformly for $|z| \leqq Z < (AT - BT - 2AB)/(A - B + 2T)$, provided $AT - BT - 2AB > 0$.*

The condition

$$\left|\frac{(t - \alpha)(z - \beta)}{(z - \alpha)(t - \beta)}\right| \geqq 1$$

for $|t| = T' < T$, $T' > B$, $|\alpha| \geqq A$, $|\beta| \leqq B$ implies

$$(38) \qquad \frac{(T' + A)(|z| + B)}{(A - |z|)(T' - B)} \geqq 1, \qquad |z| \geqq Z = \frac{AT' - BT' - 2AB}{A - B + 2T'};$$

this limit on $|z|$ is automatically less than A. Thus the circle $|t| = T'$ lies exterior to the locus R of the theorem for the assigned loci $|\alpha| \geqq A$, $|\beta| \leqq B$, $|z| \leqq Z' < Z$. The locus R is a closed region and contains the locus $|\beta| \leqq B$, hence lies interior to $|t| = T'$; the hypothesis of Theorem 11 is satisfied. Corollary 2 now follows from the fact that $T' < T$ is arbitrary, for the extreme right-hand member of (38) increases with T'.

The locus R for the assigned loci $|\alpha| \geqq A$, $|\beta| \leqq B$, $|z| \leqq Z$ is precisely the locus $|t| \leqq T'$, for R is a closed region with circular symmetry about the origin; the locus R contains the locus $|\beta| \leqq B$; by (38) the point $t = T'$ belongs to R, but no point on the axis of reals of greater modulus belongs to R.

The limit which occurs in Corollary 2 is by no means accidental. Indeed, we may set $f(z) = 1/(z + T)$, $\beta_{nk} = -B$, $\alpha_{nk} = A$. We have

$$(39) \qquad \begin{aligned} f(z) - r_n(z) &= \frac{-(A+T)^n(z+B)^{n+1}}{(T-B)^{n+1}(z+T)(z-A)^n} \\ &= \left[\frac{(A+T)(z+B)}{(T-B)(z-A)}\right]^n \frac{z+B}{(B-T)(z+T)}. \end{aligned}$$

For the value $z = (AT - BT - 2AB)/(A - B + 2T)$ we have

$$\frac{(A+T)(z+B)}{(T-B)(z-A)} = -1,$$

so the last member of (39) approaches no limit as n becomes infinite.

A generalization of Corollary 2 is obtained by the use of an arbitrary linear transformation of the complex variable.

If in Corollary 2 we make the allowable substitution $A = \infty$, we have

Corollary 3 [Méray, 1884]. *If the points β_{nk} have no limit point of modulus greater than B, and if the function $f(z)$ is analytic for $|z| < T > 2B$, then the sequence of polynomials of respective degrees n found by interpolation to $f(z)$ in the points β_{nk} converges to $f(z)$ for $|z| < T - 2B$, uniformly for $|z| \leqq Z < T - 2B$.*

If in Theorem 11 we choose R_1 as the point at infinity, inequality (36) takes the form $|z - \beta| \geqq |t - \beta|$, so we have

Corollary 4. *Let S_1 and S_2 be closed limited point sets, and let S be the closed set consisting of the closed interiors of all circles (i.e. circumferences) whose centers lie on S_1 and which pass through points of S_2. If $f(z)$ is analytic on S, and if the points β_{nk} have no limit point exterior to S_1, then the sequence of polynomials of respective degrees n defined by interpolation to $f(z)$ in the points β_{nk} converges uniformly to $f(z)$ for z on S_2.*

Corollary 3 is a special case of Corollary 4, as is also

Corollary 5. *If the points β_{nk} lie in the interval $a \leqq z \leqq b$ and if the function $f(z)$ is analytic for $|z - a| \leqq b - a$ and for $|z - b| \leqq b - a$, then the sequence of polynomials of respective degrees n defined by interpolation to $f(z)$ in the points β_{nk} converges uniformly to $f(z)$ on the interval $a \leqq z \leqq b$.*

The locus R that appears in Theorem 11 and the locus that occurs in Corollary 1 have never been studied in great detail in their dependence on R_1, R_2, and R_3. However, when R_1 is the point at infinity and R_2 and R_3 are the interiors of circles, then the locus R of Theorem 11 and the corresponding locus of Corollary 1 are bounded by Cartesian ovals; compare Walsh [1924a].

§8.10. Geometric conditions, continuation

The geometric problem which occurs in Theorem 11, that of the determination of the region R, is not symmetrically related to the four regions R_1, R_2, R_3, R, because the variables α, β, z are independent and the variable t is dependent on them, even though not uniquely determined. There naturally exist three situations other than Theorem 11 but analogous to it, where the variables α, β, z are respectively chosen as dependent. Let us study a single one of these analogous situations; the omitted situations are likewise of interest:

THEOREM 12. *Let Γ consist of a finite number of non-intersecting contours, and let S denote the complement of the locus of the point z satisfying the relation (36) when t has Γ as its locus and α and β have closed point sets S_1 and S_2 as their respective loci. Let the points α_{nk} and β_{nk} lie in S_1 and S_2 respectively. If the function $f(z)$ is analytic in closed regions bounded by Γ containing S and S_2 in their interiors, then the sequence of rational functions $r_n(z)$ of form (8) which interpolate to $f(z)$ in the points β_{nk} converges to $f(z)$ in S, uniformly on any closed subset of S.*

The relation (37) holds for t on Γ, α on S_1, β on S_2, z on S, and uniformly if z lies on any closed subset of S. The proof of Theorem 12 now follows from (12) by the method used for Theorem 11.

Theorem 12 is not merely the analogue of Theorem 11—the difference is primarily one of emphasis in the geometry of the configuration—but is also its dual in the sense that we shall indicate. Let both Γ and Λ consist of a finite number of non-intersecting contours, and let the relation (37) hold uniformly for all t in T_1, z in T_2, α on R_1, and β on R_2, where R_1 and R_2 are closed sets and where T_1 and T_2 are sets of closed regions bounded by Γ and Λ respectively. The sets T_1 and T_2 can have no common point, for when t and z coincide, the cross-ratio (t, α, z, β) takes the value unity if defined It is clear from (36) that a point of R_1 cannot belong to T_2 nor a point of R_2 to T_1; when z and α coincide and t and β are distinct, (37) cannot be satisfied; and when t and β coincide and z and α are distinct, (37) cannot be satisfied. The present configuration is entirely symmetrical with respect to t, α and z, β. If $f(z)$ is analytic in the closed complement of T_1, if the α_{nk} lie in R_1, and if the β_{nk} lie in R_2, then the sequence $r_n(z)$ of form (8) which interpolates to $f(z)$ in the points β_{nk} converges to $f(z)$ uniformly in T_2; this is essentially Theorem 11. If $g(z)$ is analytic in the closed complement of T_2, if the α_{nk} lie in R_1, and if the β_{nk} lie in R_2, then the sequence of rational functions $s_n(z)$ of respective degrees n with poles in the points β_{nk} found by interpolation to $g(z)$ in the points α_{nk} converges to $g(z)$ uniformly in T_1; this is essentially Theorem 12. From (37) we have $\alpha_{nk} \neq \beta_{nj}$.

Corollary 2 to Theorem 11 yields its own dual, in the sense that the theorem obtained from Corollary 2 by reversing the rôles of t, α and z, β can also be obtained from Corollary 2 by the transformation $w = 1/z$. The duals of Corollaries 3, 4, and 5 are neither trivial nor uninteresting, and are left to the reader for formulation.

We leave to the reader the proof (by Theorem 11 and the transformation (11)) of the analogue [Walsh, 1934b] of Theorem 11 for series of interpolation:

THEOREM 13. *Let R_1, R_2, R_3 be closed point sets, and let R be the locus of all points t when α, β, z have R_1, R_2, R_3 as their respective loci and the relation* (36) *obtains. If the function $f(z)$ is meromorphic at every point of R, if all the poles of $f(z)$ in R belong to the sequence α_k, if the points α_k have no limit point exterior to R_1, and if the points β_k are different from the α_ν and have no limit point exterior to R_2, then the formal development* (20) *of $f(z)$ converges to $f(z)$ uniformly for z on any closed set in R_3 containing no point α_k.*

The analogues of the Corollaries of Theorem 11 are corollaries of Theorem 13. Let us state the analogue of Theorem 11, Corollary 2.

COROLLARY 1. *Let the function $f(z)$ be meromorphic for $|z| < T$, let all the poles of $f(z)$ for $|z| < T$ belong to the sequence α_k which has no limit point of modulus less than A, and let the sequence $\beta_k \neq \alpha_\nu$ have no limit point of modulus greater than $B < T$. Then the formal development* (20) *converges to $f(z)$ uniformly on any closed set interior to the circle $|z| = (AT - BT - 2AB)/(A - B + 2T)$ containing no point α_k.*

The case $A = \infty$, $T = \infty$, gives

COROLLARY 2. *Let the function $f(z)$ be meromorphic at every finite point of the plane, let all the finite poles of $f(z)$ belong to the sequence $\alpha_k \to \infty$, and let the sequence $\beta_k \neq \alpha_\nu$ be uniformly limited. Then the formal development* (20) *converges to $f(z)$ uniformly on any closed limited point set containing no point α_k.*

Corollary 2 [Walsh, 1932a] is to be compared with §8.6, Ia.

The dual of Theorem 13—that is the analogue of Theorem 12 for series of interpolation—is left to the reader.

Throughout the present chapter, we have emphasized interpolation to *analytic* functions instead of to *meromorphic* functions except when we are dealing with series of interpolation. It is of course possible to study interpolation to meromorphic functions by a transformation of the same sort as (11). For instance, if the prescribed poles of the interpolating functions are $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{n,n+1}$, if the region R in which convergence is studied is finite, and if the poles of $f(z)$ in R are points α_{nk} for every n sufficiently large: $\alpha_1 = \alpha_{n1}, \alpha_2 = \alpha_{n2}, \cdots, \alpha_m = \alpha_{n,m}$, then we may replace the problem of interpolation to the function $f(z)$ by a function of form (8) by interpolation to the function (*analytic* in R)

$$(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_m) f(z)$$

in the given points of interpolation by a function of form

$$\frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{(z - \alpha_{n,m+1}) \cdots (z - \alpha_{n,n+1})}.$$

The two problems of interpolation are equivalent. Results on convergence for the modified problem lead directly to results on convergence for the proposed problem. The modified problem is the problem considered in detail in Theorems 3, 4, 5, etc.

The material of the present chapter, the study of interpolation to an analytic or meromorphic function by rational functions whose poles are prescribed, has obviously certain applications to the problem of interpolation by rational functions whose poles are not prescribed but are automatically determined by other conditions in the problem. For example, a rational function of degree n may be required to interpolate to a given function in $2n + 1$ coincident or distinct points. Fruitful methods for the study of this latter kind of problem have been developed in detail in various cases, especially for functions which are real on the axis of reals. These methods involve particularly the use of continued fractions, and deserve further study in connection with more general functions. For further details, see for instance Padé [1892], Stieltjes [1894], Nörlund [1924], and Perron [1929].

CHAPTER IX

APPROXIMATION BY RATIONAL FUNCTIONS

The present chapter is a study of convergence, degree of convergence, and best approximation by sequences of rational functions whose poles are preassigned, therefore an analogue and a generalization of Chapters IV and V. We shall need results on interpolation by rational functions supplementary to those of Chapter VIII. The case of approximation on the unit circle is of especial interest, and is treated in some detail.

§9.1. Least squares on the unit circle and interpolation

The fact (§6.1) that approximation in the sense of least squares by polynomials on the unit circle $|z| = 1$ is intimately connected with Taylor's series and therefore with interpolation suggests that approximation in the sense of least squares by more general rational functions may also be connected with interpolation in points related to the poles of those rational functions. Let us prove

THEOREM 1. *Let the function $f(z)$ be analytic on and within C: $|z| = 1$. Then the unique function of the form*

$$(1) \qquad r_n(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)}, \qquad |\alpha_k| > 1,$$

of best approximation to $f(z)$ on C in the sense of least squares, where the α_k are preassigned, is the unique function of form (1) *which interpolates to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots, 1/\bar{\alpha}_n$.*

An arbitrary function of form (1) is also an arbitrary function of form

$$(2) \qquad B_0 + \frac{B_1}{z - \alpha_1} + \frac{B_2}{z - \alpha_2} + \cdots + \frac{B_n}{z - \alpha_n},$$

except that (2) is to be suitably modified if points α_k occur multiply; the case that one or more α_k lie at infinity is exceptional in form and only in form for both (1) and (2): for (1) the corresponding factor (or factors) should simply be omitted from the denominator, and for (2) the corresponding fraction (or fractions) should be replaced by $B_k z$ (or $B_j z^2$, $B_l z^3$, etc.).

The natural method of proof of Theorem 1 is to orthogonalize (§6.2) the set of functions

$$(3) \qquad 1,\ 1/(z - \alpha_1),\ 1/(z - \alpha_2),\ \cdots,\ 1/(z - \alpha_n),$$

with the usual modification in this set for multiple or infinite α_k.* That method

* This method is carried out (with the exceptional cases omitted) by Takenaka [1925], Malmquist [1926], Walsh [1932].

is relatively unsatisfactory precisely because of the exceptional cases, which are of much importance, but which involve elaborate notation and computation. We shall therefore content ourselves with a direct verification of the theorem.

It is convenient to prove first

LEMMA I. *If the function $\phi(z)$ analytic on and within C: $|z| = 1$ vanishes in the point $z = 1/\bar{\alpha}$, then $\phi(z)$ is orthogonal on C to the function $1/(z - \alpha)$, $|\alpha| > 1$.*

The proof is immediate, for we have

$$\int_C \phi(z) \frac{|dz|}{\bar{z} - \bar{\alpha}} = \frac{i}{\bar{\alpha}} \int_C \phi(z) \frac{dz}{z - 1/\bar{\alpha}},$$

and this last expression vanishes by Cauchy's integral formula.

In precisely the same way, the reader can verify by the use of Cauchy's integral formula for the function $z^m\phi(z)$ and the formulas obtained from it by differentiation:

LEMMA II. *If the function $\phi(z)$ analytic on and within C vanishes in the origin, then $\phi(z)$ is orthogonal on C to the function 1. If $\phi(z)$ has a zero of order at least k in the point $z = 1/\bar{\alpha}$, then $\phi(z)$ is orthogonal on C to each of the functions $1/(z - \alpha)$, $1/(z - \alpha)^2, \cdots, 1/(z - \alpha)^k$, $|\alpha| > 1$.*

Lemma II is valid also in the limiting case that α is infinite: if $\phi(z)$ has a zero of order k in the origin, then $\phi(z)$ is orthogonal on C to each of the functions $1, z, \cdots, z^{k-1}$.

We are now in a position to prove Theorem 1. If $r_n(z)$ is the function of form (1) found by interpolation to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots, 1/\bar{\alpha}_n$, the difference $f(z) - r_n(z)$ vanishes in each of those points, hence is orthogonal on C to each of the functions (3), with the suitable modification in (3) for multiple and infinite α_k. It follows (§6.1, Corollary 3 to Theorem 1) that $r_n(z)$ is the linear combination of form (2) of best approximation to $f(z)$ on C in the sense of least squares. The proof is complete.

COROLLARY. *If the function $f(z)$ of Theorem 1 is not necessarily analytic on and within C but is represented interior to C by the integral*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f_1(t)\, dt}{t - z}, \tag{4}$$

where $f_1(z)$ is of class L^2 on C, then the function (1) of best approximation to $f_1(z)$ on C in the sense of least squares, where the α_k are preassigned, is the unique function (1) which interpolates to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots, 1/\bar{\alpha}_n$. In particular it is sufficient for equation (4) if $f(z)$ is of class H_2, $f_1(z) \equiv f(z)$ on C.

Let us first treat the case that $f(z)$ is of class H_2, with $f_1(z) \equiv f(z)$ on C. We remark that the equation corresponding to (4)

$$z^m f(z) = \frac{1}{2\pi i} \int_C \frac{t^m f_1(t)\, dt}{t - z}, \qquad m \geqq 0, \qquad z \text{ interior to } C,$$

is also valid; this appears directly from the relations for z interior to C

$$z^m f(z) - \frac{1}{2\pi i}\int_C \frac{t^m f_1(t)\,dt}{t - z} = \frac{1}{2\pi i}\int_C \frac{(z^m - t^m) f_1(t)\,dt}{t - z};$$

the right-hand member vanishes by virtue of §6.10 equations (45). The Corollary now follows in the present special case from the fact that in the proof of the Lemmas and of Theorem 1 we have used only Cauchy's integral formula for $\phi(z)$ and $z^m\phi(z)$, and the formulas which are obtained from it by differentiation. Those formulas all remain valid under the present hypothesis.

We now treat the general case that $f_1(z)$ is an arbitrary function of class L^2 on C. The function $f_1(z)$ can be expressed almost everywhere on C as the sum of a function of class H_2 and a function of class G_2; the latter is known (§6 11) to be orthogonal on C to every function of class H_2, hence orthogonal to each of the functions (3). Thus the original reasoning in the proof of Theorem 1 remains essentially valid also in the present case, and the Corollary is established.

If the points α_k which are not infinite are all distinct, we do not need to assume in this proof that $f_1(z)$ is of class L^2; it is sufficient if $f(z)$ is represented interior to C by (4), where $f_1(z)$ is integrable on C in the sense of Lebesgue. But the expansion of $f_1(z)$ on C is then to be interpreted merely as a formal expansion in terms of orthogonal functions; the least-square property may fail because $| f_1(z) |^2$ need not be integrable on C.

In Chapter VIII it was natural to study rational functions of form (1) rather than of form

$$(5) \qquad \frac{c_0 z^{n-1} + c_1 z^{n-2} + \cdots + c_{n-1}}{(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)}, \qquad | \alpha_k | > 1,$$

because the situation of §8.1 is essentially invariant under linear transformation. Our present problem does not have the property of invariance, and it is in order to study functions of form (5); infinite points α_k are not excluded.

To be sure, an arbitrary transformation of the form $w = (\bar{\alpha} z - 1)/(z - \alpha)$ with $| \alpha | > 1$ transforms C, its exterior, and interior, into C, its exterior, and interior respectively; a function $f(z)$ analytic on and within C is transformed into a function analytic on and within C; a rational function of degree n with poles in the points α_k is transformed into a rational function of degree n with poles in the transforms of the points α_k; interpolation in the one plane corresponds to interpolation in the other. Nevertheless, the transformation used introduces a non-trivial weight function; that is, approximation on the unit circle in the z-plane in the sense of least squares with a weight function unity corresponds to approximation on the unit circle in the w-plane in the sense of least squares with a weight function which is not identically constant. In some of our later work, we shall admit approximation with an arbitrary continuous weight function, and then the invariance is of interest. In the present study of approximation by functions of form (1), we do have invariance except for a weight function under such linear transformations; in the study of approximation by functions of form (5), the

function (5) is itself subject to the requirement of vanishing at infinity, a non-invariant property, but best approximation is characterized by interpolation in n notable points:

THEOREM 2. *Let the function $f(z)$ be analytic on and within C: $|z| = 1$, or more generally be represented interior to C by (4), where $f_1(z)$ is of class L^2. Then the function of form (5) of best approximation to $f(z)$ on C in the sense of least squares, where the α_k are preassigned, is found by interpolation to $f(z)$ in the points $1/\bar{\alpha}_k$.*

Theorem 2 follows directly from the Lemmas; the details are left to the reader.

Theorem 1 suggests a series of interpolation, if an infinite sequence of points α_k is given; the sum of the first $n + 1$ terms is the function of form (1) which is found by interpolation to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots, 1/\bar{\alpha}_n$:

$$(6) \quad f(z) = a_0 + \frac{a_1 z}{z - \alpha_1} + \frac{a_2 z(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)} + \frac{a_3 z(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z)}{(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)} + \cdots, \qquad |\alpha_k| > 1.$$

If any α_k is infinite, the factors involving that α_k are to be omitted from the denominators in (6); corresponding factors in the numerators are to be replaced by z. Each term in the right-hand member of (6) is orthogonal to all the preceding terms, as we see from the Lemmas by expressing these preceding terms in form (2). Hence, if $f(z)$ is analytic on and within C, then there are two formal expansions of $f(z)$ of form (6), the one found by interpolation to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots$, and the other found by formal expansion (as in §6.1) of $f(z)$ on C in the series of orthogonal functions.

THEOREM 3. *Let $f(z)$ be analytic on and within C: $|z| = 1$ or more generally analytic interior to C and represented interior to C by the integral (4), where $f_1(z)$ is of class L^2 on C. Then the two formal expansions of $f(z)$ of form (6) found respectively by interpolation to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots$ and by expanding $f_1(z)$ formally on C in the series (6) of orthogonal functions, are identical.*

Theorem 3 follows directly from the Lemmas. The sum $s_n(z)$ of the first $n + 1$ terms of the right-hand member of the *series of interpolation* (6) interpolates to $f(z)$ in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots, 1/\bar{\alpha}_n$, hence by the Lemmas (extended) the function $f_1(z) - s_n(z)$ is orthogonal on C to each of the functions (3), hence is orthogonal to each of the functions

$$1, \frac{z}{z - \alpha_1}, \frac{z(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)}, \cdots, \frac{z(1 - \bar{\alpha}_1 z) \cdots (1 - \bar{\alpha}_{n-1} z)}{(z - \alpha_1) \cdots (z - \alpha_n)}.$$

Consequently the series of interpolation (6) is identical with the formal expansion (6) of $f_1(z)$ on C in the series of orthogonal functions.

Theorem 3 can also be proved from the converses of Lemmas I and II, which are likewise true, or from the Corollary to Theorem 1.

Consideration of (6) as an expansion in orthogonal functions yields the

COROLLARY. *Under the conditions of Theorem 3 we have* ($n > 0$)

$$\frac{2\pi a_n}{\alpha_n\bar{\alpha}_n - 1} = \int_C f(z) \frac{\bar{z}(1 - \alpha_1\bar{z}) \cdots (1 - \alpha_{n-1}\bar{z})}{(\bar{z} - \bar{\alpha}_1) \cdots (\bar{z} - \bar{\alpha}_n)} \, | dz |,$$

$$a_n = \frac{\alpha_n\bar{\alpha}_n - 1}{2\pi i} \int_C f(z) \frac{(z - \alpha_1) \cdots (z - \alpha_{n-1}) \, dz}{z(1 - \bar{\alpha}_1 z) \cdots (1 - \bar{\alpha}_n z)};$$

in the case $\alpha_n = \infty$ *the factor* $\alpha_n\bar{\alpha}_n - 1$ *is to be replaced by unity.*

If one prefers to start not with the set of functions (3) but with the set

$$1/(z - \alpha_1), \ 1/(z - \alpha_2), \cdots, \qquad | \alpha_k | > 1,$$

one is led to a series of interpolation whose partial sums have the form (5):

$$(7) \quad f(z) = \frac{a_0}{z - \alpha_1} + \frac{a_1(1 - \bar{\alpha}_1 z)}{(z - \alpha_1)(z - \alpha_2)} + \frac{a_2(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z)}{(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)} + \cdots,$$

$$| \alpha_k | > 1 .$$

To be sure, series (7) is not a series of interpolation of the form of §8.2, equation (10); nevertheless the fundamental properties that we have established for series of interpolation hold for series (7). In fact, we have the more usual form (compare §8.2) if we write (7) as

$$(z - \alpha_1) f(z) = a_0 + a_1 \frac{1 - \bar{\alpha}_1 z}{z - \alpha_2} + a_2 \frac{(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z)}{(z - \alpha_2)(z - \alpha_3)} + \cdots.$$

The precise analogues of Theorem 3 and its Corollary hold for series of form (7).

In the remainder of the present chapter, we shall not emphasize series of interpolation. The reader will notice many applications to such series, and to the expansion of meromorphic functions, by methods used in Chapter VIII.

§9.2. Unit circle. Convergence theorems

We shall now (§§9.2–9.6) study the convergence of sequences of form (1) (where the α_k depend on n) and of series of form (6) with various conditions on the points α_k. We first apply merely geometric conditions on the given poles. Our results [Walsh, 1932] are then of considerable generality, but on the other hand our conclusions are not so strong as when more specific conditions are placed on the given poles, as in §9.5.

THEOREM 4. *Let the points* $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ *have no limit point of modulus less than* $A > 1$, *let the function* $f(z)$ *be analytic for* $| z | < T > 1$, *and let* $r_n(z)$ *be the rational function of form*

$$r_n(z) = \frac{b_{n0}z^n + b_{n1}z^{n-1} + \cdots + b_{nn}}{(z - \alpha_{n1})(z - \alpha_{n2}) \cdots (z - \alpha_{nn})}$$

of best approximation to $f(z)$ *on* C: $| z | = 1$ *in the sense of least squares. Then the sequence* $r_n(z)$ *converges to* $f(z)$ *for* $| z | < (A^2T + T + 2A)/(2AT + A^2 + 1)$, *uniformly for* $| z | \leq Z < (A^2T + T + 2A)/(2AT + A^2 + 1)$.

In the proof of Theorem 4 it will be convenient to have for reference the

LEMMA. *Under the conditions* $|t| = T > 1$, $|\alpha| \geqq A > 1$, *and* $|z| = Z > 1$, $Z < A$, *we have*

$$\left|\frac{t-\alpha}{\bar{\alpha}t-1}\right| \leqq \frac{A+T}{1+AT}, \qquad \left|\frac{\bar{\alpha}z-1}{z-\alpha}\right| \leqq \frac{AZ-1}{A-Z}. \tag{8}$$

The transformation $w = (\bar{\alpha}t - 1)/(t - \alpha)$, where α is fixed, transforms the unit circle into itself, the point $t = \infty$ into $w = \bar{\alpha}$, the point $t = \alpha$ into $w = \infty$, and the point $t = 0$ into $w = 1/\alpha$. The circle $|t| = T$ lies exterior to C, hence is transformed into a circle C' exterior to C with respect to which $w = \bar{\alpha}$ and $w = 1/\alpha$ are mutually inverse points. If $T > |\alpha|$, the circle $|t| = T$ separates the points $t = \alpha$ and $t = \infty$, so C' separates $w = \infty$ and $w = \bar{\alpha}$, hence C' is a circle containing $w = \bar{\alpha}$ in its interior. If $T < |\alpha|$, the circle $|t| = T$ does not separate the points $t = \alpha$ and $t = \infty$, so C' does not separate $w = \infty$ and $w = \bar{\alpha}$, hence C' is a circle which does not contain $w = \bar{\alpha}$ in its interior. The line in the t-plane passing through the points $t = 0, \alpha, \infty$ must correspond to a "circle" passing through the points $w = 1/\alpha, \infty, \bar{\alpha}$, that is to say, to the straight line through the origin and the point $w = \bar{\alpha}$. This line passes through the points of C' nearest to and farthest from the origin, so the points of C' which are nearest to and farthest from the origin must correspond to the points $t = T\alpha/|\alpha|$ and $t = -T\alpha/|\alpha|$. The corresponding points of C' are the points $w = (T\alpha\bar{\alpha} - |\alpha|)/(T\alpha - \alpha|\alpha|)$, $w = (T\alpha\bar{\alpha} + |\alpha|)/(T\alpha + \alpha|\alpha|)$; direct comparison shows that the latter of these is smaller in modulus, no matter whether T is greater than, less than, or equal to $|\alpha|$. Thus we have the first of the inequalities

$$\left|\frac{\bar{\alpha}t-1}{t-\alpha}\right| \geqq \frac{|\alpha|T+1}{|\alpha|+T} \geqq \frac{AT+1}{A+T};$$

the second of these inequalities may be verified by direct comparison, so the first of inequalities (8) is established.

From the discussion already given, if we use the fact proved concerning the moduli of the two points on C' which are respectively nearest to and farthest from the origin, we have the inequality

$$\left|\frac{\bar{\alpha}z-1}{z-\alpha}\right| \leqq \frac{|\alpha|Z-1}{|\alpha|-Z},$$

which implies the second of inequalities (8).

The proof of Theorem 4 is now immediate. We have for n sufficiently large

$$f(z) - r_n(z)$$
$$= \frac{1}{2\pi i}\int_\Gamma \frac{z(\bar{\alpha}_{n1}z-1)\cdots(\bar{\alpha}_{nn}z-1)(t-\alpha_{n1})\cdots(t-\alpha_{nn})f(t)\,dt}{t(\bar{\alpha}_{n1}t-1)\cdots(\bar{\alpha}_{nn}t-1)(z-\alpha_{n1})\cdots(z-\alpha_{nn})(t-z)}, \tag{9}$$

z interior to $\Gamma: |t| = T' < T$.

For n sufficiently large, we have $|\alpha_{nk}| \geqq A' < A$, $A' > 1$. The right-hand member of (9) converges uniformly to zero by virtue of (8), provided we have $|z| = Z$, $Z < A'$, $Z < T'$,

$$\frac{A' + T'}{1 + A'T'} \cdot \frac{A'Z - 1}{A' - Z} < 1,$$

that is to say, provided we have

$$Z < (A'^2T' + T' + 2A')/(2A'T' + A'^2 + 1); \tag{10}$$

this last quantity is automatically less than A' and T'. If we allow the numbers A' and T' to approach monotonically the limits A and T, the right-hand member of (10) *increases* and approaches the corresponding limit. Theorem 4 is completely proved.

The cases $A = \infty$ and $T = \infty$ are not excluded in Theorem 4. The corresponding limits on Z are respectively T and $(A^2 + 1)/(2A)$. In the former case the functions $r_n(z)$ are not necessarily polynomials; the requirement is merely that the α_{nk} have no finite limit point.

The limit which appears in Theorem 4 is by no means artificial. The conclusion is false if that limit is replaced by any larger limit, as may be seen by choosing $f(z) = 1/(z + T)$, $\alpha_{nk} = A$. We have

$$f(z) - r_n(z) = -\frac{(A + T)^n z(Az - 1)^n}{T(AT + 1)^n (z + T)(z - A)^n}. \tag{11}$$

For the particular value $z = (A^2T + T + 2A)/(2AT + A^2 + 1)$ we have

$$\frac{(A + T)(Az - 1)}{(AT + 1)(z - A)} = -1,$$

so the right-hand member of (11) approaches no limit.

COROLLARY. *Under the hypothesis of Theorem 4 we have*

$$\overline{\lim_{n\to\infty}} [\max |f(z) - r_n(z)|, z \text{ on } C]^{1/n} \leqq (A + T)/(1 + AT).$$

The Corollary follows from (8) and (9), for when z is on C we have

$$|(\bar{\alpha}_{nk}z - 1)/(z - \alpha_{nk})| = 1.$$

In the specific example considered above, we have from (11)

$$\lim_{n\to\infty} [\max |f(z) - r_n(z)|, z \text{ on } C]^{1/n} = (A + T)/(1 + AT),$$

so the result of the Corollary cannot be improved.

A relation similar to that of the Corollary is easily written down for z on the circle $|z| = Z < (A^2T + T + 2A)/(2AT + A^2 + 1)$, $Z \geqq 1$:

$$\overline{\lim_{n\to\infty}} [\max |f(z) - r_n(z)|, |z| = Z]^{1/n} \leqq \frac{A + T}{1 + AT} \cdot \frac{AZ - 1}{A - Z}.$$

The special case of Theorem 4 in which the α_{nk} do not depend on n naturally leads to a series of interpolation; compare §8.2, equation (11):

THEOREM 5. *If the function $f(z)$ is meromorphic for $|z| < T > 1$, if every pole of $f(z)$ interior to the circle $|z| = T$ occurs in the sequence $\alpha_1, \alpha_2, \cdots$ at least a number of times corresponding to its multiplicity, if the points α_k are distinct from the origin and from the points $1/\bar{\alpha}_j$, and if the points α_k have no limit point interior to the circle $|z| = A > 1$, then the formal development (6) found by interpolation in the points $0, 1/\bar{\alpha}_1, 1/\bar{\alpha}_2, \cdots$, converges to $f(z)$ for $|z| < (A^2T + T + 2A)/(2AT + A^2 + 1)$ except in the points α_k, uniformly on any closed set interior to $|z| < (A^2T + T + 2A)/(2AT + A^2 + 1)$ containing no point α_k.*

If $f(z)$ is analytic interior to C: $|z| = 1$, and if all the α_k lie exterior to C, then the sum of the first $n + 1$ terms of (6) is the rational function of form (1) of best approximation to $f(z)$ on C in the sense of least squares.

If $f(z)$ is meromorphic at every finite point of the plane, if all the finite poles of $f(z)$ occur in the sequence α_k, if the α_k are distinct from the origin and from the points $1/\bar{\alpha}_j$, and if $\lim_{k\to\infty} \alpha_k = \infty$, then (6) converges to $f(z)$ uniformly on any closed limited point set containing no α_k.

The last part of Theorem 5 is included in §8.10, Corollary 2 to Theorem 13.

In Theorems 4 and 5 we have studied interpolation and approximation by rational functions of form (1) rather than (5); the latter problem is readily handled, and yields exactly similar results.

§9.3. Unit circle. Other measures of approximation

We establish for reference the following

LEMMA I. *If $P(z)$ is a rational function of degree n whose poles lie on or exterior to the circle $|z| = \rho r > r$, and if we have*

$$|P(z)| \leqq L, \qquad \text{for } |z| = r,$$

then we have

$$|P(z)| \leqq L\left(\frac{\rho R_1 - 1}{\rho - R_1}\right)^n, \qquad \text{for } |z| = rR_1, \qquad 1 < R_1 < \rho. \tag{12}$$

We prove Lemma I in the case $r = 1$, as is sufficient. For definiteness let the poles of $P(z)$ exterior to the circle $|z| = 1$ be $\alpha_1, \alpha_2, \cdots, \alpha_m$, $m \leqq n$. The function

$$Q(z) = P(z)\,\frac{(z - \alpha_1) \cdots (z - \alpha_m)}{(1 - \bar{\alpha}_1 z) \cdots (1 - \bar{\alpha}_m z)}$$

is analytic in the extended plane for $|z| > 1$ and continuous for $|z| \geqq 1$. Thus we have $|Q(z)| \leqq L$ for $|z| = 1$, hence for $|z| \geqq 1$. The inequality $|\alpha_k| \geqq \rho$ implies by the Lemma of §9.2 for $|z| = R_1$

$$\left|\frac{1 - \bar{\alpha}_k z}{z - \alpha_k}\right| \leqq \frac{\rho R_1 - 1}{\rho - R_1},$$

from which (12) follows.

THEOREM 6. *Let the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ have no limit point of modulus less than $A > 1$, let the function $f(z)$ be analytic for $|z| < T > 1$, and let $\pi_n(z)$ be the sequence of rational functions of form*

$$(13) \qquad \pi_n(z) = \frac{b_{n0}z^n + b_{n1}z^{n-1} + \cdots + b_{nn}}{(z - \alpha_{n1})(z - \alpha_{n2}) \cdots (z - \alpha_{nn})}$$

of best approximation to $f(z)$ on C: $|z| = 1$ in the sense of Tchebycheff, in the sense of least p-th powers ($p > 0$) on C, or on the set $|z| \leq 1$ in the sense of least p-th powers ($p > 0$), in every case with a positive continuous norm function. Then the sequence $\pi_n(z)$ converges to $f(z)$ for $|z| < (A^2T + T + 2A)/(2AT + A^2 + 1)$, uniformly for $|z| \leq Z < (A^2T + T + 2A)/(2AT + A^2 + 1)$.

In every case we have

$$(14) \qquad \overline{\lim_{n\to\infty}} \, [\max |f(z) - \pi_n(z)|, z \text{ on } C]^{1/n} \leq (A + T)/(1 + AT).$$

Let $\pi_n(z)$ be the function of best approximation to $f(z)$ on C in the sense of Tchebycheff with the norm function $n(z)$, with $0 < n_1 \leq n(z) \leq N$ on C. If $q > (A + T)/(1 + AT)$ is arbitrary, we have for the rational functions $r_n(z)$ of Theorem 4 (by the Corollary to Theorem 4)

$$(15) \qquad |f(z) - r_n(z)| \leq Mq^n, \qquad z \text{ on } C.$$

Consequently we have

$$n(z)\,|f(z) - r_n(z)| \leq MNq^n, \ z \text{ on } C; \ n(z)\,|f(z) - \pi_n(z)| \leq MNq^n, \ z \text{ on } C;$$

$$(16) \qquad |r_n(z) - \pi_n(z)| \leq \frac{2MN}{n_1} q^n, \qquad z \text{ on } C.$$

For n sufficiently large the moduli $|\alpha_{nk}|$ are greater than an arbitrary $A' < A$, $A' > 1$, so from Lemma I now follows

$$|r_n(z) - \pi_n(z)| \leq \frac{2MN}{n_1} q^n \left(\frac{A'R_1 - 1}{A' - R_1}\right)^n,$$

$$\text{for } |z| \leq R_1 < A' < A, \qquad R_1 > 1.$$

Hence the sequence $r_n(z) - \pi_n(z)$ converges uniformly to the limit zero whenever we have

$$q(A'R_1 - 1)/(A' - R_1) < 1, \qquad \text{or } R_1 < (A' + q)/(1 + A'q).$$

We can allow A' to approach A and q to approach $(A + T)/(1 + AT)$; the limit of $(A' + q)/(1 + A'q)$ is $(A^2T + 2A + T)/(2AT + A^2 + 1)$. From the known regions of uniform convergence of the sequence $r_n(z)$ we can now conclude that the regions of uniform convergence of the sequence $\pi_n(z)$ are as stated. Inequality (14) follows at once from (15) and (16).

The remainder of Theorem 6 is readily proved by this same method, by the

use of the present Lemma I with $r < 1$, of §5.3 Lemma II, and of the Lemma of §5.5. The details can be supplied by the reader.

Theorem 6 is the best theorem of its kind, in the sense that the limits given on the region of convergence and degree of convergence cannot be improved without further specializing the points α_{nk} or the function $f(z)$; this follows from our remarks in connection with Theorem 4.

By inspection of the proof of Theorem 6, we can also state

COROLLARY 1. *Let the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ have no limit point of modulus less than $A > 1$, let the function $f(z)$ be analytic for $|z| < T > 1$, let C be the unit circle, and let $R_n(z)$ be a rational function of form* (13) *such that we have for a fixed q_1*

$$\overline{\lim_{n\to\infty}}\,[\max|f(z) - R_n(z)|,\, z \text{ on } C]^{1/n} \leqq q_1 \geqq (A + T)/(1 + AT), \qquad q_1 < 1 .$$

Then the sequence $R_n(z)$ converges to $f(z)$ for $|z| < (A + q_1)/(1 + Aq_1)$, uniformly for $|z| \leqq Z < (A + q_1)/(1 + Aq_1)$. Moreover, we have for $1 < Z < (A + q_1)/(1 + Aq_1)$

$$\overline{\lim_{n\to\infty}}\,[\max|f(z) - R_n(z)|,\, |z| = Z]^{1/n} \leqq q_1(AZ - 1)/(A - Z) .$$

Corollary 1 clearly applies to the functions $\pi_n(z)$ of Theorem 6.

If the analytic character of $f(z)$ is not known, we must content ourselves with a less favorable result:

COROLLARY 2. *Let the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ have no limit point of modulus less than $A > 1$, let C be the unit circle, and let $R_n(z)$ be a sequence of rational functions of form* (13) *defined for every n such that we have*

$$\overline{\lim_{n\to\infty}}\,[\max|f(z) - R_n(z)|,\, z \text{ on } C]^{1/n} \leqq q < 1 ,$$

where $f(z)$ is defined merely on C. Then the sequence $R_n(z)$ converges for $|z| < (A + q^{1/2})/(1 + Aq^{1/2})$, uniformly for $|z| \leqq Z < (A + q^{1/2})/(1 + Aq^{1/2})$, and $f(z)$ is analytic for $|z| < (A + q^{1/2})/(1 + Aq^{1/2})$.

Let $Q > q$ be arbitrary; we have for suitably chosen M

$$|f(z) - R_n(z)| \leqq MQ^n, \qquad |f(z) - R_{n+1}(z)| \leqq MQ^{n+1}, \qquad z \text{ on } C .$$

Combination of these inequalities yields

$$|R_n(z) - R_{n+1}(z)| \leqq M(1 + Q)Q^n, \qquad z \text{ on } C ,$$

whence from Lemma I

$$|R_n(z) - R_{n+1}(z)| \leqq M(1 + Q)Q^n\,[(A'R_1 - 1)/(A' - R_1)]^{2n+1} ,$$
$$\text{for } |z| = R_1 < A' < A , \qquad R_1 > 1 ,$$

provided merely n is sufficiently large; under such conditions the function $R_n(z) - R_{n+1}(z)$ is a rational function of z of degree $2n + 1$ whose poles lie exte-

rior to $|z| = A'$. Consequently the sequence $R_n(z)$ converges uniformly provided merely

$$Q^{1/2}(A'R_1 - 1)/(A' - R_1) < 1, \qquad R_1 < (A' + Q^{1/2})/(1 + A'Q^{1/2}).$$

Corollary 2 follows by letting A' and Q approach A and q respectively.

This same reasoning obviously leads to a more favorable result if the numbers $\alpha_{nk} = \alpha_k$ are independent of n, for the function $R_n(z) - R_{n+1}(z)$ is a rational function of degree only $n + 1$, whose poles lie in the points $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$:

COROLLARY 3. *Let the points $\alpha_{nk} = \alpha_k$ be independent of n, and have no limit point of modulus less than $A > 1$. Let C be the unit circle, and let $R_n(z)$ be a sequence of rational functions of form* (13) *defined for every n such that we have*

$$\overline{\lim_{n\to\infty}} [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq q < 1,$$

where $f(z)$ is defined merely on C. Then the sequence $R_n(z)$ converges for $|z| < (A + q)/(1 + Aq)$, uniformly for $|z| \leqq Z < (A + q)/(1 + Aq)$, and $f(z)$ is analytic for $|z| < (A + q)/(1 + Aq)$.

The situation of Theorem 6 is invariant under linear transformation of the form $w = (\bar{\alpha}z - 1)/(z - \alpha)$, $|\alpha| > 1$, in the sense considered in §9.1. Best approximation on the unit circle C to $f(z)$ by rational functions of degree n whose poles lie in the points α_{nk} with a positive continuous norm function, corresponds to best approximation on the unit circle in the w-plane to the function which is the transform of $f(z)$ by rational functions whose poles lie in the transforms of the points α_{nk}, with a suitably chosen positive continuous norm function. Thus we may state

COROLLARY 4. *Let the points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nk}$ have no limit point interior to $|(\bar{\alpha}z - 1)/(z - \alpha)| < A > 1$, $|\alpha| > 1$, let the function $f(z)$ be analytic for $|(\bar{\alpha}z - 1)/(z - \alpha)| < T > 1$, and let $\pi_n(z)$ be the rational function of form* (13) *of best approximation to $f(z)$ on C: $|z| = 1$ in the sense of Tchebycheff, in the sense of least p-th powers $(p > 0)$ on C, or in the region $|z| \leqq 1$ in the sense of least p-th powers $(p > 0)$, in every case with a positive continuous norm function. Then the sequence $\pi_n(z)$ converges to $f(z)$ for*

$$|(\bar{\alpha}z - 1)/(z - \alpha)| < (A^2T + T + 2A)/(2AT + A^2 + 1),$$

uniformly on any closed set interior to that circle. In every case, inequality (14) *is valid.*

Corollaries 1, 2, and 3 have their analogues in this new situation.

One further result [Walsh, 1931b] is a consequence of the method of proof of Corollary 2, and may be proved by the reader by a method entirely analogous to that of analytic extension:

COROLLARY 5. *If the sequence of rational functions $r_n(z)$ of respective degrees n converges in a region C (containing no limit points of poles of the $r_n(z)$) in such a way that we have*

$$\lim_{n\to\infty} [\max |f(z) - r_n(z)|, z \text{ on } C]^{1/n} = 0,$$

where $f(z)$ is defined merely on C, then the sequence $r_n(z)$ converges and $f(z)$ is analytic at every point P of the extended plane except the limit points of poles of the functions $r_n(z)$ and except points separated from C by such limit points. Convergence is uniform in any closed region C_1 of points P containing no such limit point, and we have also

$$\lim_{n\to\infty} [\max |f(z) - r_n(z)|, z \text{ on } C_1]^{1/n} = 0.$$

The conclusion of Corollary 5 applies in fact if C is no longer required to be a region but is a closed point set not a single point whose complement is simply connected; compare §9.7, Lemma I.

§9.4. Unit circle. Asymptotic conditions on poles

Other conditions than the geometric conditions of §§9.2 and 9.3 are of interest. We shall prove

THEOREM 7. *Let the points α_{nk} lie exterior to C: $|z| = 1$, and let the relation*

$$(17) \quad \lim_{n\to\infty} \left| \frac{(\bar\alpha_{n1} z - 1)(\bar\alpha_{n2} z - 1) \cdots (\bar\alpha_{nn} z - 1)}{(z - \alpha_{n1})(z - \alpha_{n2}) \cdots (z - \alpha_{nn})} \right|^{1/n} = |\psi(z)| \not\equiv \text{constant}$$

hold uniformly for z on an arbitrary closed subset of some region S. Let S contain C in its interior but contain in its interior no limit point of the set $1/\bar\alpha_{nk}$. Let the function $f(z)$ be analytic interior to the region R_T, where $R_{T'}$ generically denotes the region which contains in its interior C and its interior, contains in its interior no point exterior to C not in S, and is bounded by the locus $|\psi(z)| = T' > 1$ in S. Let $\pi_n(z)$ denote the function of form (13) of best approximation to $f(z)$ on C in the sense of Tchebycheff, in the sense of least p-th powers ($p > 0$), or in the region $|z| \leq 1$ in the sense of least p-th powers ($p > 0$), in every case with a positive continuous norm function. Then the sequence $\pi_n(z)$ converges to $f(z)$ in R_T, uniformly on any closed set interior to R_T.

In every case we have

$$(18) \qquad \overline{\lim_{n\to\infty}} [\max |f(z) - \pi_n(z)|, z \text{ on } C]^{1/n} \leq 1/T.$$

Let us denote by $\psi_n(z)$ the function whose modulus appears in the left-hand member of (17). By inspection we see that $\psi_n(z)$ satisfies the equation

$$\overline{\psi_n(1/\bar z)} \equiv 1/\psi_n(z).$$

It follows that whenever (17) is valid on a point set S' exterior to C, the equation also holds (if $\psi(z)$ is suitably defined and if $\psi(z) \neq 0$) on the point set which is the inverse of S' with respect to C. The function $|\psi(z)|$ satisfies the equation $|\psi(1/\bar z)| = 1/|\psi(z)|$.

The function $\psi(z)$ cannot vanish interior to S, for at each point of S the factors $|(\bar\alpha_{nk}z - 1)/(z - \alpha_{nk})|$ are uniformly bounded from zero; compare the transformation $\alpha' = (\alpha\bar z - 1)/(z - \alpha)$, where z is fixed and α and α' are variable.

The function $\log | \psi(z) |$ is the limit of a uniformly convergent sequence of harmonic functions, hence harmonic interior to S. On C we have $| \psi(z) | = 1$. Exterior to C the function $| \psi(z) |$ is the limit of a positive function which is greater than unity. The function $| \psi(z) |$ is not identically constant, so we must have $| \psi(z) | > 1$ throughout S exterior to C. It follows that the locus $| \psi(z) | = T' < T$, $T' > 1$, lies interior to R_T and exterior to C.

It will be noted that equation (17) is valid in some region whenever we have uniformly

$$(19) \qquad \lim_{n \to \infty} | (z - \alpha_{n1}) (z - \alpha_{n2}) \cdots (z - \alpha_{nn}) |^{1/n} = | \chi(z) |$$

on every closed set interior to some region Σ containing in its interior C and its interior; equation (19) holds (§7.6) with $\chi(z)$ constant whenever the α_{nk} are uniformly distributed on some curve Γ exterior to C with respect to the harmonic function conjugate to Green's function for the exterior of Γ. Equation (19) implies uniformly in the region common to Σ and its inverse in C,

$$\lim_{n \to \infty} | (\bar{\alpha}_{n1} z - 1) (\bar{\alpha}_{n2} z - 1) \cdots (\bar{\alpha}_{nn} z - 1) |^{1/n}$$

$$= \lim_{n \to \infty} | z | \cdot |(\bar{\alpha}_{n1} - 1/z) (\bar{\alpha}_{n2} - 1/z) \cdots (\bar{\alpha}_{nn} - 1/z) |^{1/n} = | z | \cdot | \chi(1/\bar{z}) | .$$

That is to say, (17) is valid uniformly if we set

$$\psi(z) \equiv z\chi(1/\bar{z})/\chi(z) .$$

For instance the α_{nk} may be the n-th roots of A^n, $A > 1$; equation (19) is valid for $| z | < A$ with $\chi(z) \equiv A$, and (17) is valid for $A > | z | > 1/A$ with $\psi(z) \equiv z$. Whenever $\chi(z)$ in (19) is identically constant, we can choose $\psi(z) \equiv z$ in (17). Whenever we can choose $\psi(z) \equiv z$, uniform convergence of the sequence $\pi_n(z)$ is in character like the uniform convergence of the Taylor development of $f(z)$, so far as points interior to S are concerned; degree of convergence of the two series corresponds if S contains the entire region of convergence of the Taylor development.

We proceed to the proof of Theorem 7. We choose first the case that $\pi_n(z) = r_n(z)$ is the function of best approximation to $f(z)$ on C in the sense of least squares with norm function unity. Convergence of this sequence $\pi_n(z)$ on the point sets as stated follows directly from (17) and from (9), where Γ is now chosen a locus $| \psi(z) | = T' < T$ in S and z lies interior to $R_{T''}$, $T'' < T'$. Moreover, for z on C we have $| (\bar{\alpha}_{nk} z - 1)/(z - \alpha_{nk}) | = 1$, so (18) follows.

By the method of proof of §9.3, Lemma I we have at once:

LEMMA II. *If $P(z)$ is a rational function all of whose poles $\alpha_1, \alpha_2, \cdots, \alpha_n$ lie exterior to C: $| z | = 1$, and if we have $| P(z) | \leqq L$ on C, then we have exterior to C*

$$|P(z)| \leqq L \left| \frac{(\bar{\alpha}_1 z - 1)(\bar{\alpha}_2 z - 1) \cdots (\bar{\alpha}_n z - 1)}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)} \right| .$$

The remainder of Theorem 7 can be proved now by the use of Lemmas I and II. We present the details only in a single case. Let us choose, for illustration and for comparison with the proof of Theorem 6, the case that $\pi_n(z)$ is the function of best approximation to $f(z)$ on C in the sense of least p-th powers. For simplicity we choose the norm function unity.

For the functions $r_n(z)$ already studied (functions of best approximation in the sense of least squares on C) we have for an arbitrary $T_1 < T$

$$(20) \qquad | f(z) - r_n(z) | \leqq M/T_1^n, \qquad z \text{ on } C.$$

There results the inequality

$$\int_C | f(z) - \pi_n(z) |^p | dz | \leqq M_1/T_1^{np},$$

or by the Lemma of §5.5,

$$(21) \qquad | f(z) - \pi_n(z) | \leqq M_2/T_1^n, \qquad | z | \leqq r < 1.$$

If we combine (20) and (21), we have

$$| \pi_n(z) - r_n(z) | \leqq M_3/T_1^n, \qquad | z | \leqq r < 1.$$

From the conditions of the theorem it follows that all the $| \alpha_{nk} |$ have a lower bound A which is greater than unity. By §9.3, Lemma I we now have ($\rho r = A$, $rR_1 = 1$)

$$(22) \qquad | \pi_n(z) - r_n(z) | \leqq \frac{M_3}{T_1^n}\left(\frac{A - r^2}{r(A - 1)}\right)^n, \qquad | z | \leqq 1,$$

where M_3 depends on r. Of course the factor $(A - r^2)/[r(A - 1)]$ is here greater than unity. By virtue of (20) and (22) we have

$$| f(z) - \pi(z) | \leqq \frac{M_4}{T_1^n}\left(\frac{A - r^2}{r(A - 1)}\right)^n, \qquad | z | \leqq 1,$$

from which follows

$$\overline{\lim_{n\to\infty}} \, [\max | f(z) - \pi_n(z) |, \ z \text{ on } C]^{1/n} \leqq \frac{1}{T_1} \cdot \frac{A - r^2}{r(A - 1)}.$$

If we allow T_1 to approach T and r to approach unity, inequality (18) results.

By virtue of (22) we may write for an arbitrary $T_2 < T$

$$| \pi_n(z) - r_n(z) | \leqq M_5/T_2^n, \qquad z \text{ on } C.$$

From Lemma II we have

$$| \pi_n(z) - r_n(z) | \leqq \frac{M_5}{T_2^n}\left| \frac{(\bar{\alpha}_{n1}z - 1)(\bar{\alpha}_{n2}z - 1) \cdots (\bar{\alpha}_{nn}z - 1)}{(z - \alpha_{n1})(z - \alpha_{n2}) \cdots (z - \alpha_{nn})} \right|, \quad z \text{ exterior to } C.$$

The locus $| \psi(z) | = T_3 < T_2$ lies interior to R_{T_2}. The uniform convergence to $f(z)$ of the sequence $\pi_n(z)$ interior to R_{T}, now follows from (17) and from the

rgence of the $r_n(z)$. The proof is complete, for $T_1 < T$

, the region S of Theorem 7 may contain the point at infinity in

7 we have supposed for simplicity that (17) is valid uniformly on an arbitrary closed subset of S; it follows that S contains no limit point of the α_{nk}. It is sufficient, however, for every α_{nk} to lie exterior to C and for (17) to be valid uniformly on any closed set interior to S except in the neighborhoods of a set of points having no limit point interior to S. In this case, the points α_{nk} may have limit points in S. The sequences $r_n(z)$ and $\pi_n(z)$ still converge to $f(z)$ interior to any R_T in which $f(z)$ is analytic, except in the limit points of the α_{nk}; convergence is uniform on any such closed set containing no such limit point. In the special case $\alpha_{nk} = \alpha_k$, the sequence $r_n(z)$ corresponds to a series of interpolation, and $f(z)$ may be meromorphic in R_T

In both Theorems 6 and 7 the conditions may be modified so that the given function $f(z)$ may be analytic merely for $|z| < 1$, continuous for $|z| \leqq 1$. Under these circumstances it is still true, and follows by the methods already used (compare §5.8), that the sequence $\pi_n(z)$ converges to $f(z)$ interior to C, uniformly on any closed point set interior to C.

In Theorems 6 and 7, the points α_{nk} and functions $r_n(z)$ and $\pi_n(z)$ need not be defined for every n.

We leave to the reader the proofs of the analogues of the Corollaries of Theorem 6:

COROLLARY 1. *Let the points α_{nk} satisfy the condition of Theorem 7, and let the function $f(z)$ be analytic interior to R_T. Let $R_n(z)$ be a rational function of form* (13) *such that we have for given T'*

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq 1/T' \geqq 1/T.$$

Then the sequence $R_n(z)$ converges to $f(z)$ interior to $R_{T'}$, uniformly on any closed set interior to $R_{T'}$. Moreover, for $T_1 < T'$ we have

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - R_n(z)|, z \text{ on } R_{T_1}]^{1/n} \leqq T_1/T'.$$

COROLLARY 2. *Let the points α_{nk} satisfy the condition of Theorem 7, and let the $R_n(z)$ be a sequence of rational functions of form* (13) *defined for every n such that we have*

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq 1/T < 1,$$

where $f(z)$ is defined merely on C. Then the sequence $R_n(z)$ converges interior to $R_{T^{1/2}}$, uniformly on any closed set interior to $R_{T^{1/2}}$, and $f(z)$ is analytic interior to $R_{T^{1/2}}$.

COROLLARY 3. *Let the points $\alpha_{nk} = \alpha_k$ be independent of n and satisfy the condition of Theorem 7. Let the $R_n(z)$ be a sequence of rational functions of form* (13) *defined for every n such that we have*

$$\overline{\lim_{n\to\infty}} \,[\max |\, f(z) - R_n(z) \,|, \, z \text{ on } C]^{1/n} \leqq 1/T < 1,$$

where $f(z)$ is defined merely on C. Then the sequence $R_n(z)$ converges interior to R_T, uniformly on any closed set interior to R_T, and $f(z)$ is analytic interior to R_T.

No new result is obtained from Theorem 7 by the transformation $w = (\bar{\alpha}z - 1)/(z - \alpha)$, $|\,\alpha\,| > 1$, for the modulus $|\,(\bar{\alpha}'z' - 1)/(z' - \alpha')\,|$ is invariant under such a transformation.

The general condition (17) will be discussed in some detail later (§9.5), but several further remarks are of interest here.

In a sense, Theorem 7 is the "best" theorem possible under the given hypothesis, for let us choose $f(z) = 1/(t - z)$, $|\,\psi(t)\,| = T$, t in S. If $r_n(z)$ denotes the function of prescribed type of best approximation to $f(z)$ on C in the sense of least squares, we have

$$f(z) - r_n(z) = \frac{z(\bar{\alpha}_{n1}z - 1) \cdots (\bar{\alpha}_{nn}z - 1)(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{t(\bar{\alpha}_{n1}t - 1) \cdots (\bar{\alpha}_{nn}t - 1)(z - \alpha_{n1}) \cdots (z - \alpha_{nn})(t - z)}.$$

For a point z in S at which $|\,\psi(z)\,| = T' > T$, we have

$$\lim_{n\to\infty} |\, f(z) - r_n(z) \,|^{1/n} = T'/T > 1,$$

so the sequence $r_n(z)$ cannot converge. For this special function $f(z)$, it is clear too that inequality (18) cannot be improved.

Under the hypothesis of Theorem 7, the function $f(z)$ may be analytic throughout S and also on the boundary of S, and yet the sequence $r_n(z)$ may converge in a region exterior to S to a value different from $f(z)$. We choose $f(z) = 1/(T - z)$, $T > 1$, and choose the α_{nk} as the n-th roots of A^n, $1 < A < T$. The region S can be chosen as $|\,z\,| < A$. Then we have

$$\begin{aligned} f(z) - r_n(z) &= \frac{z(A^n z^n - 1)(T^n - A^n)}{T(A^n T^n - 1)(z^n - A^n)(T - z)} \\ &= \frac{z[1 - 1/(A^n z^n)](1 - A^n/T^n)}{T[1 - 1/(A^n T^n)](1 - A^n/z^n)(T - z)}. \end{aligned}$$

For a value z such that $A < |\,z\,| < T$, or even $|\,z\,| > T$, this right-hand member approaches the limit $z/[T(T - z)]$

The same example shows anew the accuracy of the limit expressed by (18), even if $f(z)$ is analytic in the closed region $\bar{S}$. Inspection yields

$$\overline{\lim_{n\to\infty}} \,[\max |\, f(z) - r_n(z) \,|, \, z \text{ on } C]^{1/n} = \begin{cases} 1/T, & 1 < T < A; \\ 1/A, & 1 < A < T. \end{cases}$$

§9.5. Applications

It is now in order to mention some special cases of Theorem 7.

One fairly obvious but not uninteresting special case is that in which the numbers $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ are the first n of the numbers

$$\alpha_1, \alpha_2, \cdots, \alpha_\nu, \alpha_1, \alpha_2, \cdots, \alpha_\nu, \alpha_1, \alpha_2, \cdots, \qquad |\alpha_k| > 1.$$

Equation (17) is valid, where S consists of the extended plane except the points α_k, and we can set

$$|\psi(z)| \equiv \left| \frac{(\bar{\alpha}_1 z - 1)(\bar{\alpha}_2 z - 1) \cdots (\bar{\alpha}_\nu z - 1)}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_\nu)} \right|^{1/\nu}.$$

If the given function $f(z)$ is analytic at every point of the extended plane different from the α_k, then the sequences $r_n(z)$ and $\pi_n(z)$ converge to $f(z)$ on the entire plane except at the points α_k; this region may well be the complete domain of definition of $f(z)$. The functions $r_n(z)$ (i.e. of best approximation in the sense of least squares) can be written as the partial sums of a series of interpolation of form (6).

Another situation is somewhat similar to this. Let us suppose

$$\begin{aligned} &\alpha_1', \alpha_2', \cdots \to \alpha', \\ &\alpha_1'', \alpha_2'', \cdots \to \alpha'', \\ &\cdots\cdots\cdots\cdots\cdots, \\ &\alpha_1^{(\nu)}, \alpha_2^{(\nu)}, \cdots \to \alpha^{(\nu)}, \end{aligned}$$

where all of these numbers are in modulus greater than unity. Let the numbers $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$ be chosen as the first n of the numbers

$$(23) \qquad \alpha_1', \alpha_1'', \cdots, \alpha_1^{(\nu)}, \alpha_2', \alpha_2'', \cdots, \alpha_2^{(\nu)}, \alpha_3', \alpha_3'', \cdots.$$

Equation (17) is valid, as the reader can easily verify (compare §3.5), and we have

$$|\psi(z)| \equiv \left| \frac{(\bar{\alpha}' z - 1)(\bar{\alpha}'' z - 1) \cdots (\bar{\alpha}^{(\nu)} z - 1)}{(z - \alpha')(z - \alpha'') \cdots (z - \alpha^{(\nu)})} \right|^{1/\nu}.$$

The region S is the extended plane except the points $\alpha^{(k)}$ and the points $\alpha_j^{(k)}$. If $f(z)$ is meromorphic at every point of the extended plane other than the $\alpha^{(k)}$, if all the poles of $f(z)$ other than the $\alpha^{(k)}$ belong to the sequence (23), where each pole is counted according to its multiplicity, then the sequences $r_n(z)$ and $\pi_n(z)$ both converge to $f(z)$ at every point of the extended plane except the points $\alpha^{(k)}$ and the points $\alpha_j^{(k)}$; this region of convergence may well be the complete domain of definition of $f(z)$. The functions $r_n(z)$ are the partial sums of a series of interpolation (6).

Whenever the points $\alpha_{nk} = \alpha_k$ are independent of n and of modulus greater than unity, the functions $r_n(z)$ are the partial sums of a series of form (6). Under these conditions suppose the points α_{nk} satisfy the conditions of Theorem 7, and let us assume purely as a matter of convenience that S and the α_n are bounded. Then an obvious consequence of (17) is

$$\lim_{n\to\infty} \left| \frac{z(1 - \bar{\alpha}_1 z)(1 - \bar{\alpha}_2 z) \cdots (1 - \bar{\alpha}_{n-1} z)}{(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)} \right|^{1/n} = |\psi(z)|, \qquad z \neq 0,$$

for z in S, uniformly on any closed set interior to S not containing the origin. Consider an arbitrary series of form (6); let us set $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 1/T'$. We assume $T' > 1$; in the contrary case the series converges at no point of S exterior to C. It follows then from §3.4, Theorem 5 that (6) converges uniformly on any closed set interior to $R_{T'}$ and diverges at every point of S exterior to $R_{T'}$ if $R_{T'}$ exists; the sum $f(z)$ of the series is analytic throughout the interior of $R_{T'}$. If $R_{T'}$ does not exist, the function $f(z)$ is analytic interior to every existent $R_{T_1'}$, $T_1' < T'$. We have

$$\overline{\lim_{n\to\infty}} \, [\max |f(z) - s_n(z)|, z \text{ on } C]^{1/n} \leqq 1/T',$$

where $s_n(z)$ is the sum of the first $n + 1$ terms of the series. Of course this series (6) is the unique formal expansion of $f(z)$ found by interpolation, hence is also the formal expansion of $f(z)$ on C in the series of orthogonal functions.

Under the hypothesis of Theorem 7 with $\alpha_{nk} = \alpha_k$ suppose that $f(z)$ is analytic interior to R_T but has a singularity on the boundary of R_T. Let us set $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = 1/T'$, where a_n is defined as in (6). The inequality $T' > T$ would imply uniform convergence of the development (6) in some region containing R_T in its interior, which is impossible. The inequality $T' < T$ would imply divergence of (6) exterior to $R_{T'}$, hence at some point interior to R_T, in contradiction to Theorem 7. Thus we have $T' = T$; series (6) diverges in S exterior to R_T. If $f(z)$ is analytic interior to every R_T, then we have

$$\overline{\lim_{n\to\infty}} \, |a_n|^{1/n} \leqq 1/T_0,$$

where T_0 is the least upper bound of all T for which R_T is defined. Of course the function $\psi(z)$ may be defined exterior to S, but a locus $|\psi(z)| = T'$ wholly or partly exterior to S has no significance in the present connection.

Theorem 7 has application also to the case of an *arbitrary* region D which contains in its interior both C and its interior, and which satisfies certain broad requirements. Let us suppose for the moment that D is finite and is bounded by a finite number of non-intersecting analytic Jordan curves. Let S be the region common to D and to the inverse of D in C. Then S is also bounded by a finite number of analytic Jordan curves We shall use the method of §8.7.

Let S_1 denote the subregion of D exterior to C. Let $U(x, y)$ be the function which takes on the value zero on the boundary of D and the value $+1$ on C, is continuous in $\bar{S}_1$ and harmonic in S_1. Then $U(x, y)$ can be extended harmonically by reflection across C, so as to be harmonic throughout the interior of S, and takes on the value -1 on the boundary of S interior to C.

If the points α_{nk} are uniformly distributed with respect to the harmonic function conjugate to $U(x, y)$ on the boundary of D and the points β_{nk}' on C (by the method of §8 7) we have

$$\lim_{n\to\infty} \left| \frac{(z - \beta_{n1}')(z - \beta_{n2}') \cdots (z - \beta_{nn}')}{(z - \alpha_{n1})(z - \alpha_{n2}) \cdots (z - \alpha_{nn})} \right|^{1/n} = \exp[\sigma U(x, y)], \tag{24}$$

uniformly for z on any closed set interior to S_1, where the constant σ is suitably chosen.

If we set $z = 1/\bar{z}'$ and make use of the fact $|\beta'_{nk}| = 1$, we have

$$|(z - \beta'_{nk})/(z - \alpha_{nk})| = |(z' - \beta'_{nk})/(\bar{\alpha}_{nk}z' - 1)|,$$

$$(25)\quad \lim_{n\to\infty} \left| \frac{(z' - \beta'_{n1}) \cdots (z' - \beta'_{nn})}{(\bar{\alpha}_{n1}z' - 1) \cdots (\bar{\alpha}_{nn}z' - 1)} \right|^{1/n} = \exp[\sigma U'(x', y')], \qquad z' = x' + iy',$$

for z' on any closed point set interior to S_2, the subregion of S interior to C; in (25) the arguments (x', y') of the function U' are naturally those which correspond under inversion in C: $z = 1/\bar{z}'$ to those in (24). We define U' by the equation $U(x, y) = U'(x', y')$.

For z on any closed set interior to S_2, the limit of the left-hand member of (24) exists (compare §§8.7 and 8.8) uniformly and is a non-vanishing constant. For z' on any closed set interior to S_1, the limit of the left-hand member of (25) exists uniformly and is a non-vanishing constant. From (24) and (25) we now have for suitably chosen $V(x, y)$

$$(26)\qquad \lim_{n\to\infty} \left| \frac{(\bar{\alpha}_{n1}z - 1) \cdots (\bar{\alpha}_{nn}z - 1)}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \right|^{1/n} = V(x, y),$$

uniformly for z on any closed set interior to S_1 or interior to S_2. The actual computation of $V(x, y)$ is not difficult here; compare §§9.11 and 9.12. It is sufficient for our present purposes to notice that $V(x, y)$ is constant on the boundary of D. Since the points α_{nk} lie on the boundary of D and the points $1/\bar{\alpha}_{nk}$ lie on the boundary of the inverse of D, the logarithm of the left-hand member of (26) is harmonic and bounded from zero on any closed set interior to S. Hence equation (26) is valid uniformly for z on any closed set interior to S, and is precisely of form (17). We shall point out later (§9.11, Theorem 18) that $V(x, y)$ cannot be identically constant.

If there is given a region D which contains in its interior C and its interior but is no longer bounded by a finite number of non-intersecting analytic Jordan curves, let S_1 denote the subregion of D exterior to C. The left-hand member of (26) is invariant under transformation of the form $w = (\bar{\alpha}z - 1)/(z - \alpha)$, $|\alpha| > 1$, so we can assume the point at infinity not to lie interior to D. Assume that there exists a function $U(x, y)$ harmonic interior to S_1, continuous in $\bar{S}_1$, which takes on the value unity on C, and which takes on the value zero on the boundary of D. By the methods of §§4.3, 4.4, and 7.6, there can be chosen points β'_{nk} on C and points α_{nk} on the boundary of D if C is bounded by a finite number of non-intersecting Jordan curves and otherwise approaching the boundary of D, such that (24) is valid for z on any closed set interior to S_1. Similarly we can satisfy (25) interior to S_2, hence (26) uniformly for z on any closed set interior both to D and to the inverse of D in C. The points α_{nk} may also be chosen as points α_k independent of n.

Equation (26) is now sufficient for the application of Theorem 7; our restric-

tions on the region D are quite light, and involve merely the existence of a certain harmonic function. If the region D is here the complete domain of definition of the function $f(z)$, then the sequence $r_n(z)$ represents $f(z)$ throughout its entire domain of definition.

It is worth remarking, in connection with the study of the region D bounded by Jordan curves, that the properties expressed by (18) and uniform convergence of the sequence $\pi_n(z)$ interior to every $R_{T'}$, $T' < T$, are also possessed by the functions of the sequence $r_n(z)$ studied in §8.7, Theorem 9 for the present region S_1 and found by interpolation in the points $\beta'_{n+1,k}$ on C. In fact, the present function $V(x, y)$ is constant on the boundary of D exterior to C and is constant on C; its logarithm is harmonic in the region S_1. It is immaterial in §8.7 whether we start with that harmonic function $U(x, y)$ or a harmonic function of the form $aU(x, y) + b$ not identically constant, where a and b are constants; the same uniformly distributed points α_{nk} and β_{nk} present themselves. In particular, for the present region S_1 one can use the harmonic function $\log V(x, y)$. Of course in Theorem 7 a sequence of functions of best approximation can be written down immediately (namely the sequence for least squares), so it is unnecessary to use the functions of §8.7; but other situations (e.g. §9.10) are less simple; in these other situations it is frequently desirable to have at hand sequences with known and strong convergence properties. As a matter of fact, it is true that the points $1/\bar{\alpha}_{nj}$ themselves are uniformly distributed on the boundary of S with respect to the harmonic function $U_1(x, y)$ conjugate to $U(x, y)$. This follows from the fact that the points α_{nj} are uniformly distributed on the boundary of D with respect to $U_1(x, y)$, and from the fact that each level curve of $U_1(x, y)$ is its own inverse (is anallagmatic) with respect to the circle C. Our present use of (26) is thus essentially an application of §8.7, Theorem 9.

§9.6. Poles with limit points on circumference

In our study of convergence of rational functions of best approximation in the sense of least squares on C with preassigned poles, Theorems 4 and 7 represent certain sufficient conditions for convergence, derived by means of (9). Equation (9), or what amounts essentially to the same thing, the corresponding expansion of $1/(t - z)$, furnishes the natural basis for the study of the problem. When z lies on C certain factors under the integral sign in (9) are of modulus unity, so actual convergence to the given function of the sequence of best approximation would seem to depend on the approach to zero of the factor

$$\frac{(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{(\bar{\alpha}_{n1} t - 1) \cdots (\bar{\alpha}_{nn} t - 1)}, \qquad |t| > 1.$$

We proceed to elaborate this remark, and shall take up the general study of approximation on the unit circle when the prescribed poles are still exterior to C but may have limit points on C itself.

THEOREM 8. *Let the points α_{nk} exterior to $C: |z| = 1$ be given, and let $f(z)$ be an arbitrary function analytic on and within C. Let $r_n(z)$ denote the function*

of form (13) *of best approximation to* $f(z)$ *on* C *in the sense of least squares. A necessary and sufficient condition that for every such* $f(z)$ *we have*

$$(27) \qquad \lim_{n\to\infty} r_n(z) = f(z), \qquad |z| < 1,$$

is

$$(28) \qquad \lim_{n\to\infty} \left| \frac{(t-\alpha_{n1})(t-\alpha_{n2})\cdots(t-\alpha_{nn})}{(\bar\alpha_{n1}t-1)(\bar\alpha_{n2}t-1)\cdots(\bar\alpha_{nn}t-1)} \right| = 0,$$

uniformly for $|t| \geqq T > 1$.

If this condition is satisfied, we have

$$(29) \qquad \lim_{n\to\infty} r_n(z) = f(z), \text{ uniformly for } |z| \leqq r < 1,$$

not merely whenever $f(z)$ *is analytic on and within* C, *but more generally whenever* $f(z)$ *is of class* H_2. *Whenever* $f(z)$ *is analytic on and within* C, *we have*

$$(30) \qquad \lim_{n\to\infty} r_n(z) = f(z), \qquad \text{uniformly for } |z| \leqq 1.$$

If we make the substitution $\bar z = 1/t$, and replace the corresponding function by its conjugate, equation (28) assumes the equivalent form

$$(31) \qquad \lim_{n\to\infty} \left| \frac{(\bar\alpha_{n1}z-1)(\bar\alpha_{n2}z-1)\cdots(\bar\alpha_{nn}z-1)}{(z-\alpha_{n1})(z-\alpha_{n2})\cdots(z-\alpha_{nn})} \right| = 0,$$

uniformly for $|z| \leqq r < 1$.

The mere existence of the limit in (28) implies the uniformity as stated, for each function $(t-\alpha_{nk})/(\bar\alpha_{nk}t-1)$ is analytic in the closed region exterior to C and in that closed region is of modulus not greater than unity. Hence the functions

$$f_n(t) \equiv \frac{(t-\alpha_{n1})\cdots(t-\alpha_{nn})}{(\bar\alpha_{n1}t-1)\cdots(\bar\alpha_{nn}t-1)}$$

are uniformly limited exterior to C and form a normal family exterior to C; the uniformity of convergence in (28) follows from the convergence.

If $f(z)$ is analytic interior to C and continuous on and within C, condition (28) implies not merely (27) but (29); we see this from (9) and (31), where in (9) we identify Γ with C; we note that $|(t-\alpha_{nk})/(\bar\alpha_{nk}t-1)| = 1$ for t on C. Even if $f(z)$ is of class H_2 equations (9) and (29) are valid; indeed, it is sufficient if $f(z)$ is merely represented by Cauchy's integral provided the α_{nk} are all distinct, except that $r_n(z)$ is then to be defined by interpolation instead of by least-square properties on C. If $f(z)$ is analytic on and within C, condition (28) implies (30); this follows from (9) and (28), if the curve Γ of (9) is chosen as a circle containing C in its interior, such that $f(z)$ is analytic on and within Γ.

It remains to establish the necessity of (28). We need prove merely the

convergence in (28) without the uniformity, which we do by choosing $f(z) \equiv 1/(t - z)$, $|t| > 1$. We have

$$f(z) - r_n(z) = \frac{z(\bar{\alpha}_{n1} z - 1) \cdots (\bar{\alpha}_{nn} z - 1)(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{t(\bar{\alpha}_{n1} t - 1) \cdots (\bar{\alpha}_{nn} t - 1)(z - \alpha_{n1}) \cdots (z - \alpha_{nn})(t - z)}.$$

For the particular value $z = 1/\bar{t}$, our hypothesis (27) yields

$$\lim_{n\to\infty} \left| \frac{(t - \alpha_{n1}) \cdots (t - \alpha_{nn})}{(\bar{\alpha}_{n1} t - 1) \cdots (\bar{\alpha}_{nn} t - 1)} \right|^2 = 0,$$

and this implies (28).

It is worth noticing that if the limit in (28) is zero even for a single value of t, say t_0, $|t_0| > 1$, not a limit point of the α_{nk}, then condition (28) is fulfilled uniformly as stated. Indeed, the functions $f_n(t)$ defined above form a normal family exterior to C. Every limit function of the family vanishes at t_0, hence by Hurwitz's theorem vanishes identically. That is to say, the sequence $f_n(t)$ converges to zero exterior to C, hence converges uniformly for $|t| \geqq T > 1$. In particular (this remark is due jointly to Shen and the present writer) if the α_{nk} are uniformly limited, a necessary and sufficient condition for (28) is ($t_0 = \infty$)

$$\lim_{n\to\infty} |\alpha_{n1}\alpha_{n2} \cdots \alpha_{nn}| = \infty.$$

It is not true that (28) as stated is necessarily a consequence of having the limit in (28) zero at a single value of t, $|t| > 1$, say t_0, if t_0 is a limit point of the α_{nk}. We illustrate this fact by an example. Choose the distinct points t_0 and t_1 both exterior to C, and let us set

$$|(t_1 - t_0)/(\bar{t}_0 t_1 - 1)| = \lambda > 0, \qquad \lambda < 1.$$

Choose α_{n1} near t_0 so that we have

$$|(t_0 - \alpha_{n1})/(\bar{\alpha}_{n1} t_0 - 1)| < 1/n, \qquad |(t_1 - \alpha_{n1})/(\bar{\alpha}_{n1} t_1 - 1)| > \lambda/2;$$

then choose α_{nk}, $k > 1$, near C so that we have

$$\left| \frac{(t_1 - \alpha_{n1}) \cdots (t_1 - \alpha_{nn})}{(\bar{\alpha}_{n1} t_1 - 1) \cdots (\bar{\alpha}_{nn} t_1 - 1)} \right| > \frac{\lambda}{2}.$$

The inequality

$$\left| \frac{(t_0 - \alpha_{n1}) \cdots (t_0 - \alpha_{nn})}{(\bar{\alpha}_{n1} t_0 - 1) \cdots (\bar{\alpha}_{nn} t_0 - 1)} \right| < \frac{1}{n}$$

holds. The limit in (28) is zero for $t = t_0$, and cannot be zero for $t = t_1$.

Theorem 8 has immediate application to other measures of best approximation:

Corollary 1. *If the function $f(z)$ is analytic interior to C, continuous for $|z| \leqq 1$, if condition (28) is fulfilled, and if $\pi_n(z)$ is the sequence of rational functions of form (13) of best approximation to $f(z)$ on C in the sense of Tchebycheff, on*

C in the sense of least p-th powers $(p > 0)$, or on $|z| \leqq 1$ in the sense of least p-th powers $(p > 0)$, in every case with a positive continuous norm function, then the sequence $\pi_n(z)$ converges uniformly to $f(z)$ for $|z| \leqq r < 1$. For best approximation in the sense of Tchebycheff, convergence is uniform for $|z| \leqq 1$.

Let $\epsilon > 0$ be given. By §2.4, Theorem 5, there exists a polynomial $p(z)$ such that we have

$$|f(z) - p(z)| \leqq \epsilon, \qquad |z| \leqq 1.$$

Let $p_n(z)$ denote the function of form (13) of best approximation to $p(z)$ on C in the sense of Tchebycheff. The expression

$$\mu_n = \max |p(z) - p_n(z)|, \qquad z \text{ on } C,$$

is not greater than the corresponding expression where $p_n(z)$ is replaced by the function of form (13) of best approximation to $p(z)$ on C in the sense of least squares. Then by Theorem 8 equation (30) we have $\lim_{n\to\infty} \mu_n = 0$. If $\pi_n(z)$ denotes the function of form (13) of best approximation to $f(z)$ on C in the sense of Tchebycheff, we now have for z on C

$$\begin{aligned} \max |f(z) - \pi_n(z)| &\leqq \max |f(z) - p_n(z)| \\ &\leqq \max |f(z) - p(z)| + \max |p(z) - p_n(z)|, \end{aligned}$$

$$\varlimsup_{n\to\infty} [\max |f(z) - \pi_n(z)|, \quad z \text{ on } C] \leqq \epsilon.$$

Then we can write

$$\lim_{n\to\infty} [\max |f(z) - \pi_n(z)|, \quad z \text{ on } C] = 0,$$

so the sequence $\pi_n(z)$ converges uniformly to $f(z)$ for $|z| \leqq 1$.

If $\pi_n(z)$ is now the function (13) of best approximation to $f(z)$ on C in the sense of Tchebycheff with a norm function, or in some other sense, it follows from what has just been proved that the measure of approximation of $\pi_n(z)$ to $f(z)$ approaches zero with $1/n$. Our conclusion follows from the Lemma of §5.5 and from Lemma II of §5.3.

In Theorem 8 we have considered $r_n(z)$ as the function of form (13) of best approximation to $f(z)$ on C in the sense of least squares. A similar result is

COROLLARY 2. *Let the points α_{nk} exterior to C: $|z| = 1$ be given, and let $f(z)$ be an arbitrary function analytic on and within C. Let $r_n(z)$ denote the function of form* (13) *of best approximation to $f(z)$ on C in the sense of Tchebycheff. Then* (28) *is a necessary and sufficient condition that for every such function $f(z)$ equation* (30) *should be valid.*

The sufficiency of condition (28) is already established; let us prove the necessity. Equation (30) implies that for the sequence of rational functions (13) of best approximation to $f(z)$ on C in the sense of least squares the measure of ap-

proximation approaches zero. Equation (27), where the $r_n(z)$ are the functions of best approximation to $f(z)$ on C in the sense of least squares, follows from §5.8, Theorem 16; equation (28) now results from Theorem 8.

Condition (28) is essentially the condition for the divergence to zero of the infinite product

$$\prod_{n=1}^{\infty}\left[\prod_{k=1}^{n}\frac{t-\alpha_{nk}}{\bar{\alpha}_{nk}t-1}\cdot\prod_{k=1}^{n-1}\frac{\bar{\alpha}_{n-1,k}t-1}{t-\alpha_{n-1,k}}\right],$$

where the second product in the bracket is to be taken as unity for $n = 1$. Various conditions for the divergence to zero of such an infinite product are of course well known. Let us consider in detail such a condition which involves merely the moduli $|\alpha_{nk}|$.

THEOREM 9. *Let the numbers $A_{nk} > 1$ be given. Equivalent necessary and sufficient conditions that for every choice of the α_{nk}, with $|\alpha_{nk}| = A_{nk}$ finite or infinite, equation (28) be valid are*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{A_{nk}-1}{A_{nk}} = \infty, \tag{32}$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{T+A_{nk}}{A_{nk}T+1}\right) = 0, \qquad T > 1,\ T \textit{ finite}. \tag{33}$$

Condition (33) is independent of T, as we shall see. Let us first prove the equivalence of (32) and (33). From the condition $A_{nk} > 1$ we may write

$$\frac{T-1}{T+1} \leqq \frac{(T-1)(A_{nk}-1)/(A_{nk}T+1)}{(A_{nk}-1)/A_{nk}} \leqq \frac{T-1}{T}, \qquad T > 1.$$

It follows that (32) is equivalent to the condition

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(T-1)(A_{nk}-1)}{A_{nk}T+1} = \infty. \tag{34}$$

For $0 < x \leqq X < 1$ the condition

$$mx < \log(1-x) < Mx, \qquad m < 0,\ M < 0,$$

is satisfied for suitably chosen values of m and M depending on X. The quantity

$$(T-1)(A_{nk}-1)/(A_{nk}T+1)$$

is positive and not greater than $(T-1)/T$, so condition (34) is equivalent to

$$\lim_{n\to\infty}\sum_{k=1}^{n}\log[1-(T-1)(A_{nk}-1)/(A_{nk}T+1)]$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}\log[(T+A_{nk})/(A_{nk}T+1)] = -\infty;$$

this last condition is equivalent to (33), which is therefore equivalent to (32) and is independent of T.

If the numbers A_{nk} are uniformly limited, condition (32) can be written in the simpler form

$$\lim_{n\to\infty} \sum_{k=1}^{n} (A_{nk} - 1) = \infty .$$

The general condition (32) is satisfied, as can be proved easily, provided we have $A_{nk} \geqq A_n > 1$ and provided the two conditions

$$\lim_{n\to\infty} n(A_n - 1) = \infty, \qquad \lim_{n\to\infty} A_n^n = \infty,$$

(which are equivalent to each other) are satisfied. We proceed with the proof of the theorem.

Let (33) be satisfied. By the Lemma of §9.2 we have

$$|(t - \alpha_{nk})/(\bar{\alpha}_{nk}t - 1)| \leqq (|t| + A_{nk})/(A_{nk}|t| + 1),$$

so that (33) implies (28) without the uniformity, hence with the uniformity as stated.

Reciprocally, let (28) be satisfied for every choice of t and of the α_{nk}, $|\alpha_{nk}| = A_{nk}$. We choose $t = -T$, $\alpha_{nk} = A_{nk}$, so that (28) takes the form of (33). Theorem 9 is completely proved.

It is worth remarking that equation (33) implies (28) merely under the condition $|\alpha_{nk}| \geqq A_{nk}$, as follows again by the Lemma of §9.2. Furthermore, the condition $|\alpha_{nk}| \geqq A_{nk}$ implies

$$0 < (A_{nk} - 1)/A_{nk} \leqq (|\alpha_{nk}| - 1)/|\alpha_{nk}|,$$

so (32) implies

$$\lim_{n\to\infty} \sum_{k=1}^{n} (|\alpha_{nk}| - 1)/|\alpha_{nk}| = \infty .$$

That is to say, in Theorem 9 the condition $|\alpha_{nk}| = A_{nk}$ can be replaced by the weaker condition $|\alpha_{nk}| \geqq A_{nk}$, in deriving (28) from either (32) or (33).

Equations (32) and (33) clearly may even be equivalent to (28), for a given set of numbers α_{nk}, $|\alpha_{nk}| = A_{nk}$. For instance, we may choose the α_{nk} as the n-th roots of $-A_n^n$, $A_n > 1$, A_n bounded. Condition (28) can be written

$$\lim_{n\to\infty} (t^n + A_n^n)/(A_n^n t^n + 1) = 0, \qquad \text{for every } |t| > 1. \tag{35}$$

When we set $t = T$, this condition becomes

$$\lim_{n\to\infty} (T^n + A_n^n)/(A_n^n T^n + 1) = 0, \tag{36}$$

which in turn implies (35) by virtue of the Lemma of §9.2. Condition (33) here takes the form

$$\lim_{n\to\infty} [(T + A_n)/(A_n T + 1)]^n = 0 ,$$

which like (36) is equivalent to the condition $\lim_{n\to\infty} A_n^n = \infty$.

The conditions of Theorems 8 and 9 are applicable even in the case that $\alpha_{nk} = \alpha_k$ is independent of n. Condition (28) is then equivalent to the divergence of the infinite product

$$\prod_{k=1}^{\infty} | (t - \alpha_k)/(\bar{\alpha}_k t - 1) | ,$$

a condition to be studied in more detail in Chapter X.

§9.7. General point sets; degree of convergence

The extension of the results of §§9.2 – 9.5 to regions D more general than circles is not easy. When D is a circle and poles α_{nk} lie exterior to D, and when the given function $f(z)$ is analytic in the closed region D, a particular set of rational functions (13) of best approximation (i.e., for least squares) can be written down directly; this set enables us to study the convergence of the sequence of rational functions (13) when best approximation is measured in any of a large number of other ways. When D is no longer a circle, it is still possible to orthogonalize the functions (3) with respect to the boundary of D, and thereby to study convergence of the sequence (13) of best approximation in the sense of least squares. The formulas are so involved, however, that simple results on overconvergence and degree of convergence are not readily obtainable from them.

In a most interesting paper, Shen [1935] has been able to define points of interpolation which yield sequences (13) with a sufficiently high degree of convergence and with sufficiently great overconvergence to enable us to study in a satisfactory manner the convergence of functions of best approximation. The method is the natural generalization of Shen's method for polynomials (§7 8)

THEOREM 10. *Let C be a closed limited point set not a single point whose complement K is simply connected. Let $w = \phi(z)$ denote a function which maps K onto $|w| > 1$ so that the points at infinity correspond to each other. Let C_R denote generically the curve $|\phi(z)| = R > 1$ in K. Let the points α_{nk}, $k = 1, 2, \cdots, n$; $n = 0, 1, 2, \cdots$ be given with no limit points interior to C_A. Then there exist points β_{nk} on C such that for every function $f(z)$ analytic on and within C_T the rational function $r_n(z)$ of form (13) found by interpolation to $f(z)$ in the points β_{nk} satisfies the conditions*

$$\overline{\lim_{n\to\infty}} [\max | f(z) - r_n(z) | , z \text{ on } C]^{1/n} \leq (A + T)/(AT + 1) , \tag{37}$$

$$\lim_{n\to\infty} r_n(z) = f(z) , \tag{38}$$

uniformly for z on and within C_Z, $Z < (A^2T + T + 2A)/(2AT + A^2 + 1)$.

Before proceeding with the proof of Theorem 10, it will be convenient to have for reference two lemmas.

LEMMA I. *Let C, $\phi(z)$ and C_R be as in Theorem 10. Let the points α_{nk} lie exterior to C, and let $P(z)$ be a function analytic in the extended plane exterior to C except for possible poles in the points α_{nk}, no such pole of greater multiplicity than the number of times the corresponding α_{nk} appears in the set. Let us suppose*

$$|P(z)| \leqq L, \qquad z \text{ on } C, \tag{39}$$

in the sense that every limit value of $|P(z)|$ when z approaches the boundary of C from the exterior is less than or equal to L. Then for z exterior to C we have

$$|P(z)| \leqq L \left| \frac{[\bar{\phi}(\alpha_{n1})\,\phi(z) - 1] \cdots [\bar{\phi}(\alpha_{nn})\phi(z) - 1]}{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]} \right|. \tag{40}$$

In particular, if all the points α_{nk} lie on or exterior to C_A, then for z on C_Z, $Z < A$, we have

$$|P(z)| \leqq L[(AZ - 1)/(A - Z)]^n. \tag{41}$$

Inequality (40) is immediate, for the function

$$P(z) \frac{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]}{[\bar{\phi}(\alpha_{n1})\,\phi(z) - 1] \cdots [\bar{\phi}(\alpha_{nn})\,\phi(z) - 1]}$$

is analytic exterior to C; its modulus is single-valued and continuous in K; of course the obvious modification is to be made here and in similar formulas below if any α_{nk} is infinite. This modulus is not greater than L on C, in the sense used in connection with (39), hence is not greater than L in K; this implies (40). Inequality (41) follows from (40) by the Lemma of §9.2, if we set $w = \phi(z)$.

LEMMA II. *Let C, $\phi(z)$ and C_R be as in Theorem 10. Let the points α_{nk} lie exterior to C, and let $P(z)$ be a function analytic exterior to C which vanishes in each of the points α_{nk}. Let us suppose (39) valid in the sense of Lemma I. Then for z exterior to C we have*

$$|P(z)| \leqq L \left| \frac{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]}{[\bar{\phi}(\alpha_{n1})\,\phi(z) - 1] \cdots [\bar{\phi}(\alpha_{nn})\,\phi(z) - 1]} \right|.$$

In particular, if all the points α_{nk} lie on or exterior to C_A, then for z on C_T we have

$$|P(z)| \leqq L\,[(A + T)/(1 + AT)]^n.$$

The proof is immediate, by consideration of the modulus of the function

$$P(z) \frac{[\bar{\phi}(\alpha_{n1})\,\phi(z) - 1] \cdots [\bar{\phi}(\alpha_{nn})\,\phi(z) - 1]}{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]},$$

and by further application of the Lemma of §9.2.

To prove Theorem 10 we introduce the notation

$$V_n(z_1, z_2, \cdots, z_{n+1}) \equiv \frac{\displaystyle\prod_{\substack{j,k=1\\ j<k}}^{n+1} (z_j - z_k)}{\displaystyle\prod_{j=1,k=1}^{j=n+1,k=n} (z_j - \alpha_{nk})};$$

this is a continuous function of the $n + 1$ independent variables z_j on C; a possible factor $z_j - \alpha_{nk}$ corresponding to an infinite α_{nk} is simply to be omitted. The modulus of this function therefore has a maximum, necessarily positive, for the z_j on C; we suppose this maximum taken on for the arguments $z_j = \beta_{nj}$. The points β_{nj} are not necessarily uniquely determined, but any particular determination is satisfactory for the present purpose. The points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{n,n+1}$ are clearly all distinct.

Let t be an arbitrary point of C_{R_1}, $R_1 < A$. Choose n so large that the points α_{nk} lie exterior to C. Denote by $r_n(t, z)$ the function of form (13) which interpolates to $1/(t - z)$ in the points β_{nj}. We have

$$r_n(t,z) = \sum_{j=1}^{n+1} \frac{1}{t - \beta_{nj}} \frac{V_n(\beta_{n1}, \cdots, \beta_{n,j-1}, z, \beta_{n,j+1}, \cdots, \beta_{n,n+1})}{V_n(\beta_{n1}, \cdots, \beta_{n,n+1})}, \tag{42}$$

as we shall proceed to verify. For fixed t, the function $r_n(t, z)$ defined by (42) has as its only poles in z the points α_{nk}, with not more than the prescribed multiplicities; when z takes on the value β_{nj}, the function $r_n(t, z)$ so defined takes on the value $1/(t - \beta_{nj})$ as it should do, the function $r_n(t, z)$ is a rational function of z of degree n. These conditions determine $r_n(t, z)$ uniquely

The modulus of the last fraction in (42) is not greater than unity for z on C, by the very choice of the points β_{nj}, so we may write for z on C and t on C_{R_1}

$$| r_n(t, z) | \leqq (n + 1)/d_1, \tag{43}$$

where d_1 is the minimum distance from C to C_{R_1}. By virtue of §8.1, equation (3) we have

$$\frac{1}{t - z} - r_n(t, z) = \frac{1}{t - z} \frac{\omega_n(z)}{\omega_n(t)}, \qquad \omega_n(z) \equiv \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})}.$$

If we take (43) into account, we now have for z on C and t on C_{R_1}

$$\left| \frac{1}{t - z} \frac{\omega_n(z)}{\omega_n(t)} \right| \leqq \frac{n + 2}{d_1},$$

$$\left| \frac{\omega_n(z)}{\omega_n(t)} \right| \leqq \frac{(n + 2)d_2}{d_1}, \tag{44}$$

where d_2 denotes the maximum distance from a point of C to a point of C_{R_1}.

For z fixed on C, we consider $\omega_n(z)/\omega_n(t)$ as a function of t on and exterior to C_{R_1} and apply Lemma II. The function which maps the exterior of C_{R_1} onto $|w| > 1$ is here $\phi(z)/R_1$; for n sufficiently large the points α_{nk} lie on or exterior to $C_{A_1} = (C_{R_1})_{A_1/R_1}$, $A_1 < A$, so inequality (44) yields for z on C and for t on $C_{T_1} = (C_{R_1})_{T_1/R_1}$, $T_1 > R_1$,

$$\left|\frac{\omega_n(z)}{\omega_n(t)}\right| \leqq \frac{(n+2)d_2}{d_1}\left(\frac{R_1(T_1+A_1)}{A_1T_1+R_1^2}\right)^n. \tag{45}$$

These preliminaries complete, we can now choose $T_1 > T$ so that $f(z)$ is analytic on and within C_{T_1}, then choose $A_1 < A$ so near A and $R_1 > 1$ so near unity that we have

$$R_1(T_1 + A_1)/(A_1T_1 + R_1^2) < (T + A)/(AT + 1)\,; \tag{46}$$

this last number is necessarily greater than $(T_1 + A)/(AT_1 + 1)$. The familiar equation for interpolation is here chosen in the form

$$f(z) - r_n(z) = \frac{1}{2\pi i}\int_{C_{T_1}} \frac{\omega_n(z)\, f(t)\, dt}{\omega_n(t)\,(t - z)}, \qquad z \text{ on } C. \tag{47}$$

Inequality (37) follows from (45) and (46).

From Lemma I and inequality (45) we may write for n sufficiently large

$$\left|\frac{\omega_n(z)}{\omega_n(t)}\right| \leqq \frac{(n+2)d_2}{d_1}\left(\frac{R_1(T_1+A_1)}{A_1T_1+R_1^2}\right)^n\left(\frac{A_1Z-1}{A_1-Z}\right)^n$$

for z on C_Z, $Z < A_1$, and for t on C_{T_1}. Equation (47) remains valid and implies the uniform convergence of $r_n(z)$ to $f(z)$ on and within C_Z provided

$$\frac{R_1(T_1+A_1)}{A_1T_1+R_1^2}\cdot\frac{A_1Z-1}{A_1-Z} < 1,$$

with $Z < T_1$, and this condition is satisfied for suitably chosen $A_1 < A$ and for suitably chosen $R_1 > 1$ provided

$$\frac{A+T}{AT+1}\cdot\frac{AZ-1}{A-Z} < 1, \qquad \text{or} \qquad Z < \frac{A^2T + T + 2A}{2AT + A^2 + 1}.$$

This last inequality implies $Z < A$ and $Z < T$, so Theorem 10 is established.

§9.8. General point sets; best approximation

Our application of Theorem 10 to sequences of best approximation depends largely on the following theorem [Walsh, 1932c]:

THEOREM 11. *Let C be a closed limited point set not a single point whose complement K is simply connected, and let $\phi(z)$ and C_R have the usual significance. Let the points α_{nk} have no limit point interior to C_A, let the function $f(z)$ be defined*

on C, and let there exist a sequence $r_n(z)$ of functions of form (13) *such* *have*

(48) $$\overline{\lim_{n\to\infty}} [\max |f(z) - r_n(z)|, z \text{ on } C]^{1/n} \leqq (A + T)/(1 + AT),$$

(49) $$\lim_{n\to\infty} r_n(z) = f(z), \text{ uniformly for } z \text{ on any closed set interior to } C_Z,$$

where $Z = (A^2T + T + 2A)/(2AT + A^2 + 1)$.

If the functions $R_n(z)$ of form (13) *satisfy the condition*

(50) $$\overline{\lim_{n\to\infty}} [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq (A + T)/(1 + AT),$$

then we have also

(51) $$\lim_{n\to\infty} R_n(z) = f(z), \text{ uniformly for } z \text{ on any closed set interior to } C_Z,$$

where $Z = (A^2T + T + 2A)/(2AT + A^2 + 1)$.

Let $A_1 < A$ and $T_1 < T$ be arbitrary, with $A_1 > 1$, $T_1 > 1$. By virtue of (48) and (50) we can write for z on C if M is suitably chosen

$$|f(z) - r_n(z)| \leqq M\left(\frac{A_1 + T_1}{1 + A_1T_1}\right)^n, \qquad |f(z) - R_n(z)| \leqq M\left(\frac{A_1 + T_1}{1 + A_1T_1}\right)^n,$$

$$|r_n(z) - R_n(z)| \leqq 2M\left(\frac{A_1 + T_1}{1 + A_1T_1}\right)^n.$$

The function $r_n(z) - R_n(z)$ is a rational function of degree n whose poles lie in the points α_{nk}. By Lemma I we can write

$$|r_n(z) - R_n(z)| \leqq 2M\left(\frac{A_1 + T_1}{1 + A_1T_1}\right)^n \left(\frac{A_1Z_1 - 1}{A_1 - Z_1}\right)^n, \qquad z \text{ on } C_{Z_1}, Z_1 < A_1.$$

Whenever Z_1 is less than Z, we have $Z_1 < A$ and we can choose $A_1 < A$ and $T_1 < T$ such that

$$\frac{A_1 + T_1}{1 + A_1T_1} \frac{A_1Z_1 - 1}{A_1 - Z_1} < 1,$$

with $Z_1 < A_1$, whence (51) follows from (49).

By inspection of this proof we can formulate

COROLLARY 1. *Under the conditions of Theorem* 11 *we may write*

$$\overline{\lim_{n\to\infty}} [\max |f(z) - R_n(z)|, z \text{ on } C_S]^{1/n} \leqq \frac{A + T}{1 + AT} \frac{AS - 1}{A - S}, \qquad S < Z,$$

provided the corresponding relation holds for the functions $r_n(z)$.

By Theorem 10 we may with Shen state

COROLLARY 2. *In Theorem* 11 *if the function* $f(z)$ *is known to be analytic interior to* C_T, *then functions* $r_n(z)$ *exist so that the conditions* (48) *and* (49) *are fulfilled.*

Of course Theorem 10 yields only

$$\varlimsup_{n\to\infty} [\max |f(z) - r_n(z)|, z \text{ on } C]^{1/n} \leqq (A + T_1)/(1 + AT_1),$$

where $T_1 < T$ is arbitrary; if T_1 is allowed to approach T, inequality (48) follows. Under the conditions of Corollary 2, the hypothesis of Corollary 1 is also satisfied.

We leave to the reader the proof of the following; the method is precisely that of Theorem 6, Corollaries 2 and 3:

COROLLARY 3. *Let* C, $\phi(z)$ *and* C_R *have the same significance as in Theorem* 10, *let the points* α_{nk} *have no limit point interior to* C_A, *and let* $R_n(z)$ *be a sequence of rational functions of form* (13) *defined for every* n *such that we have*

$$\varlimsup_{n\to\infty} [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq q < 1,$$

where the function $f(z)$ *is defined merely on* C. *Then the sequence* $R_n(z)$ *converges uniformly for* z *on and within every* C_{S_1}, $S_1 < S = (A + q^{1/2})/(1 + Aq^{1/2})$, *and* $f(z)$ *is analytic interior to* C_S.

COROLLARY 4. *Let* C, $\phi(z)$ *and* C_R *have the same significance as in Theorem* 10, *let the poles* $\alpha_{nk} = \alpha_k$ *be independent of* n *and have no limit point interior to* C_A, *and let* $R_n(z)$ *be a sequence of rational functions of form* (13) *defined for every* n *such that we have*

$$\varlimsup_{n\to\infty} [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq q < 1,$$

where $f(z)$ *is defined merely on* C. *Then the sequence* $R_n(z)$ *converges uniformly for* z *on and within every* C_{S_1}, $S_1 < S = (A + q)(1 + Aq)$, *and* $f(z)$ *is analytic interior to* C_S.

The most general known result on sequences of best approximation where the poles are restricted as in Theorem 10 is due to Shen:

THEOREM 12. *Let* C *be a closed limited point set not a single point whose complement* K *is simply connected, and let* $\phi(z)$ *and* C_R *have the usual significance. Let the points* α_{nk} *have no limit point interior to* C_A, *and let the function* $f(z)$ *be analytic interior to* C_T. *Let* $R_n(z)$ *denote the function of form* (13) *of best approximation to* $f(z)$ *on* C *in the sense of Tchebycheff, in the sense of least* p*-th powers* ($p > 0$) *over the boundary (assumed rectifiable,* C *consisting of a finite number of Jordan arcs or regions or both) of* C, *in the sense of least* p*-th powers over the area of* C *(if* C *is a closed region), in the sense of least* p*-th powers over* γ: $|w| = 1$ *when* K *is mapped onto the exterior of* γ *so that the points at infinity correspond to each other,*

or in the sense of least p-th powers over γ: $|w| = 1$ *or over the region* $|w| \leqq 1$ *when* C *is mapped onto* $|w| \leqq 1$ *(C being a closed region), in every case with a positive continuous norm function. Then in each case both* (50) *and* (51) *are valid.*

The restriction that K be connected can be lightened here in various cases; compare Chapter V, especially §5.3. Likewise the requirement that the norm function be positive and continuous can be lightened.

If $R_n(z)$ is the function of best approximation in the sense of Tchebycheff, the conclusions (50) and (51) are direct consequences of Theorem 11 Corollary 2 and Theorem 11 respectively. In the study of other measures of approximation, we need Lemma I of §9.7 and several other lemmas [Walsh, 1931b]; in each case C_R has the usual meaning.

LEMMA III. *Let C be a closed limited point set whose complement is simply connected and whose boundary* Γ *has positive linear measure. If* $P(z)$ *is a rational function of degree n whose poles lie on or exterior to* C_A, *and if we have*

$$\int_\Gamma |P(z)|^p\, |dz| \leqq L^p, \qquad p > 0,$$

then we have

$$|P(z)| \leqq LL'[(AR - 1)/(A - R)]^n, \qquad z \text{ on } C_R, \qquad R < A,$$

where L' *depends on* R *but not on* $P(z)$.

LEMMA IV. *Let C be a closed limited point set not a single point, whose complement is simply connected. Let the complement of C be mapped onto* $|w| > 1$ *so that the points at infinity correspond to each other. If* $P(z)$ *is a rational function of degree n whose poles lie on or exterior to* C_A, *and if we have*

$$\int_\gamma |P(z)|^p\, |dw| \leqq L^p, \qquad p > 0,$$

γ: $|w| = 1$, *then we have*

$$|P(z)| \leqq LL'[(AR - 1)/(A - R)]^n, \qquad z \text{ on } C_R, \qquad R < A,$$

where L' *depends on* R *but not on* $P(z)$.

In Lemma IV the values of $P(z)$ on γ are to be chosen as the values obtained in the w-plane by normal approach to γ. The proof of Lemmas III and IV is left to the reader. These Lemmas have the same relation to the Lemmas of §§5.2 and 5.4 respectively as Lemma I of §9.7 has to the Lemma of §4.6, so the new proofs are not difficult. In Lemma III if Γ is not a Jordan curve we use the results of Seidel [1934].

The complete proof of Theorem 12 is likewise not difficult. Applications of the Lemmas in the various measures of approximation are similar, so we study in detail only one further example, that of approximation in the sense of least p-th

powers measured over the area of the region C. For simplicity we choose the norm function unity.

By Theorem 10 we have for arbitrary $q > (A + T)/(1 + AT)$ and for suitably chosen M

$$\iint_C | f(z) - r_n(z) |^p dS \leqq Mq^{np},$$

so by the definition of the functions $R_n(z)$ we have

$$\iint_C | f(z) - R_n(z) |^p dS \leqq Mq^{np}.$$

Combination of these two inequalities yields (by §5.2, inequalities (10))

$$\iint_C | R_n(z) - r_n(z) |^p dS \leqq M_1q^{np}.$$

If C' denotes an arbitrary closed point set interior to C we have (§5.3, Lemma II)

$$| R_n(z) - r_n(z) | \leqq M_2q^n,$$

where M_2 depends on C' but not on n.

Choose C' as a Jordan region interior to C, and denote by $w = \phi'(z)$ the function which maps the complement of C' onto $| w | > 1$ so that the points at infinity correspond to each other. Denote generically by C'_R the curve $| \phi'(z) | = R > 1$. On the boundary of C we have $| \phi(z)/\phi'(z) | < 1$, so this inequality holds at every point of K and every curve C'_R lies interior to the corresponding curve C_R. In particular the points α_{nk} lie exterior to C_A, hence exterior to C'_A, for n sufficiently large. By Lemma I we have for z on C'_Q, $Q < A$, and for n sufficiently large

$$(52) \qquad | R_n(z) - r_n(z) | \leqq M_2q^n[(AQ - 1)/(A - Q)]^n.$$

For fixed but arbitrary $Q > 1$, the curve C'_Q lies exterior to C provided C' is suitably chosen (§2.1, Theorem 2), so inequality (52) holds for z on C.

By Theorem 10 and by (52) for z on C we can write

$$\overline{\lim_{n\to\infty}} \, [\max | f(z) - R_n(z) |, \quad z \text{ on } C]^{1/n} \leqq q(AQ - 1)/(A - Q).$$

If we allow q to approach $(A + T)/(1 + AT)$ and allow Q to approach unity, we obtain (50) and then obtain (51) by Theorem 11.

The reader should have no difficulty in considering in detail the other cases of Theorem 12. In each case the conditions of Corollary 1 to Theorem 11 are fulfilled.

Of course the case that all α_{nk} lie at infinity is not excluded in Theorem 12; we set $A = \infty$. Theorem 12 thus yields many of the results of Chapter V.

By a transformation $z' = 1/(z - \alpha)$, α in K, the following situation reduces to that of Theorem 12; suitable modifications are to be made in the transformed

norm functions; the situations of Theorems 10 and 11 can be similarly transformed:

COROLLARY. *Let C be a closed limited point set not a single point whose complement K is simply connected, let the point $z = \alpha$ lie in K, and let $C_R(\alpha)$ denote the locus $|\phi(z, \alpha)| = R > 1$ in K, where the function $w = \phi(z, \alpha)$ maps K onto the region $|w| > 1$ so that the point $z = \alpha$ corresponds to the point $w = \infty$. For the present purposes, we denote generically by the "interior" of $C_R(\alpha)$ the complement of the set $|\phi(z, \alpha)| \geqq R > 1$. Let the points α_{nk} have no limit point in the "interior" of $C_A(\alpha)$, and let the function $f(z)$ be analytic in the "interior" of $C_T(\alpha)$. Let $R_n(z)$ denote the function of form* (13) *of best approximation to $f(z)$ on C in the various senses as in Theorem* 12, *each with a positive continuous norm function. Then in each case we have* (50), *and we also have* (51) *uniformly for z on any closed set in the "interior" of $C_Z(\alpha)$, $Z = (A^2T + T + 2A)/(2AT + A^2 + 1)$.*

§9.9. Extensions

The limits that appear in (50) and in connection with (51) are clearly the most favorable ones possible if we are dealing with limits common to all possible point sets C of the sort contemplated in Theorem 12, for those limits are precisely the limits that present themselves when C is a circle (§9.2). Much more can be asserted, however,—*given C, A, and T, the limits that appear in* (50) *and in connection with* (51) *are for each measure of approximation the best possible limits which hold for all choice of the α_{nk} and of $f(z)$.* Let z_0 be an arbitrary point of the curve C_T. We choose $f(z) = 1/(z - z_0)$, and choose all the α_{nk} equal to α, where $\phi(\alpha) = -A\phi(z_0)/T$; then α lies on C_A. Best approximation to $f(z)$ on C by rational functions of z whose poles lie in the points $z = \alpha$ is equivalent to best approximation with a suitable norm function after transforming to the ζ-plane, setting $\zeta = 1/(z - \alpha)$. Under this transformation, let C, z_0, and $f(z)$ correspond to C', z_0' and $f'(z)$ respectively. Best approximation on C corresponds to best approximation on C' by *polynomials* in ζ. This sequence of polynomials converges maximally on C', hence can converge uniformly throughout no neighborhood of the point $\zeta_1 = 1/(z_1 - \alpha)$, where z_1 is defined by

$$\phi(z_1) = \frac{\phi(\alpha)}{A} \frac{A^2T + T + 2A}{2AT + A^2 + 1};$$

the point ζ_1 lies on the curve C_R' on which z_0 lies, and we have $R = (1 + AT)/(A + T)$. For the functions $R_n(z)$ of best approximation, inequality (50) becomes an equality, hence cannot be improved. The geometric relations between the points α, z_0, and z_1 become more transparent (compare §9.2, proof of Lemma) if the exterior of C is mapped onto the region $|w| > 1$ so that the points at infinity correspond to each other. Such transformation does not alter the relationships existing between the various mapping functions for the exterior of C, whether the poles of the mapping functions lie at $z = \alpha$ or $z = \infty$. The

choice of points z_0 and z_1 just made does not differ greatly from the choice we have made elsewhere in specific examples.

We prefer not to use the term *maximal convergence* in connection with (50) and (51), for that term has previously been used in connection with approximating polynomials for a *specific* function rather than a class of functions. It is clear that both (50) and (51) can be improved for certain specific functions $f(z)$ and points α_{nk} which satisfy the given conditions; compare the case $\alpha_{nk} = \alpha$ on C_A, with $f(z) = 1/(z - z_0)$, $\phi(z_0) = T\phi(\alpha)/A$.

Let C and the points α_{nk} be given as in Theorem 10. Shen [1935] has derived necessary and sufficient conditions on points β_{nk} having no limit point exterior to C, in order that (37) and (38) should hold for all functions $f(z)$ analytic interior to C_T. Two such conditions (equivalent to each other) are

$$\lim_{n\to\infty} M_n^{1/n} = 1,$$

$$M_n = \max \left| \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \phi(\alpha_{n1}) \cdots \phi(\alpha_{nn}) \right|, \qquad z \text{ on } C;$$

$$\lim_{n\to\infty} \left| \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \cdot \frac{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]\, \phi(\alpha_{n1}) \cdots \phi(\alpha_{nn})}{[\bar{\phi}(\alpha_{n1})\, \phi(z) - 1] \cdots [\bar{\phi}(\alpha_{nn})\, \phi(z) - 1]} \right|^{1/n} = 1,$$

for z exterior to C, uniformly on any closed limited set exterior to C. These conditions are entirely analogous to those derived in Chapter VII, and the analogous methods can be used in the proofs.

The problem of studying convergence in an arbitrary region C by rational functions whose poles are prescribed and lie exterior to C but have limit points on the boundary of C has also been studied by Shen in a paper as yet unpublished.

The analogues of Theorems 10, 11, and 12, where now the complement K of C is multiply connected, are somewhat more complicated but scarcely more difficult. Under the hypothesis of Theorem 10, if the function $w = \phi(z)$ maps K onto $|w| > 1$ so that the points at infinity correspond to each other, then the function

$$w = [\bar{\phi}(\alpha)\phi(z) - 1]/[\phi(z) - \phi(\alpha)]$$

maps K onto $|w| > 1$ so that the point $z = \alpha$ in K corresponds to $w = \infty$. There exists no such simple relation when K is multiply connected, for both mapping functions are multiple valued. In our proofs of Theorems 10–12 we have largely used single-valued mapping functions; those theorems can be directly extended to the case that K is multiply connected if the locus C_A is defined in terms of the mapping function but if the new locus C_R, $R = (AT + 1)/(A + T)$, is defined no longer in terms of the original mapping function, but is defined now from C_A with the use of the auxiliary mapping function under which arbitrary

points α on C_A correspond to the point at infinity. One may readily include results here on approximation when the α_{nk} are required to have no limit point interior to a given locus C_A, or are required to have no limit point except on a given closed set D exterior to C. For the details, one may consult Shen [1935] and Walsh [1932c].

The problem of approximation on a point set which separates the plane can be treated by these same methods. Let us give a simple but typical illustration [Walsh, 1932c].

LEMMA. *Let Γ be an arbitrary Jordan curve of the z-plane in whose interior the origin lies. If $r_{2n}(z)$ is a function of the form*

$$(53) \qquad r_{2n}(z) = a_{-n}z^{-n} + a_{-n+1}z^{-n+1} + \cdots + a_0 + a_1 z + \cdots + a_n z^n ,$$

and if we have for z on Γ

$$| r_{2n}(z) | \leqq M ,$$

then we have also

$$(54) \qquad | r_{2n}(z) | \leqq MR^n, \; R > 1,$$

for z on $\bar{\Gamma}_R$. Here $\bar{\Gamma}_R$ denotes the closed region between and bounded by the two curves $| \phi(z) | = R$ and $| \Phi(z) | = R$, where $w = \phi(z)$ maps the exterior of Γ onto $| w | > 1$ so that $z = \infty$ corresponds to $w = \infty$, and $w = \Phi(z)$ maps the interior of Γ onto $| w | > 1$ so that $z = 0$ corresponds to $w = \infty$.

The function

$$r_{2n}(z)[\phi(z)]^{-n}$$

is analytic exterior to Γ, continuous in the corresponding closed region. Its modulus on Γ is not greater than M, so its modulus on $| \phi(z) | = R$ is also not greater than M. That is to say, inequality (54) is valid on $| \phi(z) | = R$. Similarly we derive (54) for z on $| \Phi(z) | = R$. The function $r_{2n}(z)$ is analytic in the closed region $\bar{\Gamma}_R$; inequality (54) is valid on the boundary, hence valid in the closed region.

With the same notation, we shall prove

THEOREM 13. *If $f(z)$ is analytic in $\bar{\Gamma}_R$, then there exist rational functions $r_{2n}(z)$ of respective forms (53) such that we have for z on Γ*

$$(55) \qquad | f(z) - r_{2n}(z) | \leqq M/R^n ,$$

where M is independent of n and of z.

If there exist rational functions $r_{2n}(z)$ of form (53) such that (55) is valid for z on Γ, where $f(z)$ is defined merely on Γ, then the sequence $r_{2n}(z)$ converges interior to Γ_R, uniformly on any closed set interior to Γ_R, so $f(z)$ is analytic throughout the interior of Γ_R.

If $f(z)$ is given analytic in $\bar{\Gamma}_R$, it can be split up into two components (§1.7) analytic respectively on and within $|\phi(z)| = R$, and on and exterior to $|\Phi(z)| = R$. Suitable approximation of these components by polynomials in z and polynomials in $1/z$ respectively yields (55), where $r_{2n}(z)$ is the sum of the approximating polynomials.

The second part of Theorem 13 follows easily. We have from (55)

$$|r_{2n+2}(z) - r_{2n}(z)| \leqq (M + M/R)/R^n, \qquad z \text{ on } \Gamma,$$

and from the Lemma

$$|r_{2n+2}(z) - r_{2n}(z)| \leqq (M + M/R)R_1^n/R^n, \qquad z \text{ on } \bar{\Gamma}_{R_1}, \quad R_1 < R.$$

The conclusion follows.

In the second part of Theorem 13 it is also true that the function $r_{2n}(z)$ can be split up into components, and these components converge on C to the respective components of $f(z)$ like a geometric series of ratio $1/R_1$, $R_1 < R$. This fact may be proved from the results on degree of convergence just established, by using Cauchy's integral for $f(z) - r_{2n}(z)$ over the two curves $|\phi(z)| = R_2 > 1$ and $|\Phi(z)| = R_2$ respectively, where R_2 is near unity.

Theorem 13, without the fact of overconvergence, was proved by de la Vallée Poussin [1919] for the case that Γ is a circle whose center is the origin.

Theorem 13 has obvious application to the study of convergence of rational functions $r_{2n}(z)$ of best approximation in the sense of Tchebycheff; if $f(z)$ is analytic interior to Γ_R but has a singularity on the boundary, then the sequence of rational functions $r_{2n}(z)$ of best approximation to $f(z)$ on Γ in the sense of Tchebycheff converges to $f(z)$ interior to Γ_R, uniformly on any closed set interior to Γ_R. Similar results follow from Theorem 13 and from the methods used in Chapter V concerning rational functions $r_{2n}(z)$ of best approximation on Γ, in the sense of Tchebycheff with norm function, in the sense of least p-th powers ($p > 0$, and Γ rectifiable), and in the sense of least p-th powers ($p > 0$) measured over the circle $|w| = 1$ when either the interior or exterior of Γ is mapped onto $|w| > 1$. A large number of extensions (for instance to the case where Γ is no longer a Jordan curve but is a closed set which has no interior point but separates the plane into precisely two regions, or where Γ is a closed region that separates the plane into two regions) now lie immediately at hand.

We state one further theorem that follows from the general theory of approximation by rational functions on point sets whose complements are not necessarily simply connected or even connected; the proof can be given from §2.8, Theorem 15, by the method used in proving Theorem 8, Corollary 1:

THEOREM 14. *Let C be a closed point set whose boundary B consists of a finite number of Jordan arcs or curves or both, such that B separates the plane into no more than a finite number of regions. The complement K of C consists of a finite number of regions $C_1, C_2, \cdots, C_\nu$. Let points α_{nk} be given in K such that the number of points α_{nk} that lie in each C_j becomes infinite with n, and such that the α_{nk} have*

no limit point on C. If the function $f(z)$ is analytic in the interior points of C, continuous on C, then there exists a sequence of functions of form (13) *such that we have*

$$\lim_{n\to\infty} r_n(z) = f(z)$$

uniformly for z on C.

If the α_{nk} have asymptotically the same number of points distributed among the various regions into which C separates the plane, and if $f(z)$ is analytic on C, then there exist functions $r_n(z)$ of form (13) *converging to $f(z)$ geometrically on C. Reciprocally, if there exist functions $r_n(z)$ of form* (13) *converging to $f(z)$ geometrically on C, then $f(z)$ is analytic on C.*

The latter part of this theorem is true even if we omit the restriction that B should consist of a finite number of Jordan arcs or curves, provided each of the finite number of regions into which B separates the plane is of finite connectivity and regular.

§9.10. General point sets; asymptotic conditions on poles

It is obviously not the purpose in Theorems 4–6 or Theorems 10–12 to show the advantage over best approximation by polynomials of best approximation by rational functions when the poles α_{nk} are given; indeed, Theorems 4–6 and 10–12 are in some respects less precise and the properties derived are less favorable for many purposes than the corresponding results (Chapter V) on best approximation by polynomials. It is rather our purpose in Theorems 4–6 and 10–12 to study very general situations, with relatively weak hypotheses and necessarily weak conclusions. However, if further properties of the given poles α_{nk} are known, stronger conclusions can be proved and more favorable properties deduced, properties going far beyond corresponding properties for approximation by polynomials. This fact has already been exemplified in Theorem 7 by assuming asymptotic conditions on the α_{nk}, and will now be further exemplified in a similar way.

Shen did not study convergence under the hypothesis of asymptotic conditions on the α_{nk}, but his methods apply with appropriate modifications. Let us prove

THEOREM 15. *Let C be a closed limited point set not a single point whose complement K is simply connected, and let $\phi(z)$ have the usual significance. Let the points α_{nk} be given exterior to C and let the relation*

$$(56)\quad \lim_{n\to\infty}\left|\frac{[\bar{\phi}(\alpha_{n1})\,\phi(z)-1]\cdots[\bar{\phi}(\alpha_{nn})\,\phi(z)-1]}{[\phi(z)-\phi(\alpha_{n1})]\cdots[\phi(z)-\phi(\alpha_{nn})]}\right|^{1/n} = |\psi(z)| \not\equiv \text{constant}$$

hold uniformly on an arbitrary closed subset of some region S. Let $S + C$ be a region which contains C in its interior but contains in its interior no limit point of the α_{nk}. Let R_T denote generically the closed region (if existent) which contains C in its interior, contains no point not in C or S, and is bounded by the locus

$|\psi(z)| = T > 1$ *in* S. *If the function* $f(z)$ *is analytic in the closed region* R_T, *then there exists a sequence of functions* $r_n(z)$ *of form* (13) *such that we have*

$$\varlimsup_{n\to\infty} [\max |f(z) - r_n(z)|, \quad z \text{ on } C]^{1/n} < 1/T, \tag{57}$$

$$\lim_{n\to\infty} r_n(z) = f(z), \tag{58}$$

uniformly interior to R_T.

Indeed, we shall prove the existence of a set of points β_{nk} on the boundary of C such that (57) and (58) hold for the sequence $r_n(z)$ found by interpolation in those points; the points β_{nk} are precisely the points introduced by Shen.

The present proof is slightly more complicated than those of §§9.4 and 9.7, due to the fact that the function $\psi(z)$ need not be defined on C, hence that (56) cannot be used on C itself. However, the points α_{nk} have no limit point interior to S, hence for n sufficiently large lie exterior to some C_A: $|\phi(\alpha_{nk})| > A$. On C_Z, $Z < A$, we have $|\phi(z)| = Z$, therefore by the Lemma of §9.2,

$$1 \leqq \left| \frac{[\bar\phi(\alpha_{n1})\,\phi(z) - 1] \cdots [\bar\phi(\alpha_{nn})\,\phi(z) - 1]}{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]} \right| \leqq \left(\frac{AZ - 1}{A - Z}\right)^n,$$

so on C_Z we have

$$1 \leqq |\psi(z)| \leqq (AZ - 1)/(A - Z).$$

Thus $|\psi(z)|$ approaches unity as Z approaches unity, or as z in S approaches C. The function $|\psi(z)|$ when defined so as to have the value unity at every point of the boundary of C is continuous at every such point. At every point of S we have $|\psi(z)| \geqq 1$, hence by the Principle of Maximum $|\psi(z)| > 1$.

Of course condition (56) takes precisely the form (17), if we make the substitution $w = \phi(z)$. The implications of (56) are much easier to deduce by studying the w-plane; compare §§9.4 and 9.5. In particular, equation (56) apparently is not in invariant form, for the definition of $\phi(z)$ involves the point at infinity. That lack is apparent rather than real, as the reader may prove; compare §9.4. Condition (56) is precisely the same for two sets C and C' whose complements can be mapped on one another so that the prescribed points α_{nk} in the two planes correspond to each other.

Let us denote generically by L_R the locus $|\psi(z)| = R > 1$. Choose n so large that the points α_{nk} lie exterior to C. Then for z on C and t on L_R we have as in §9.7

$$|r_n(t, z)| \leqq (n + 1)/d_1,$$

where d_1 is the minimum distance from C to L_R. Similarly we find for z on C and t on L_R,

$$\left|\frac{\omega_n(z)}{\omega_n(t)}\right| \leqq \frac{(n + 2)d_2}{d_1}, \qquad \omega_n(z) \equiv \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})}, \tag{59}$$

where d_2 denotes the maximum distance from a point of C to a point of L_R. By the method of proof of §9.7, Lemma II we now have for z on C and t on or exterior to L_R,

$$(60)\qquad \left|\frac{\omega_n(z)}{\omega_n(t)}\right| \leqq \frac{(n+2)d_2\,\mu_n}{d_1}\left|\frac{[\phi(t)-\phi(\alpha_{n1})]\cdots[\phi(t)-\phi(\alpha_{nn})]}{[\bar\phi(\alpha_{n1})\,\phi(t)-1]\cdots[\bar\phi(\alpha_{nn})\,\phi(t)-1]}\right|,$$

$$\mu_n = \max\left[\left|\frac{[\bar\phi(\alpha_{n1})\,\phi(t)-1]\cdots[\bar\phi(\alpha_{nn})\,\phi(t)-1]}{[\phi(t)-\phi(\alpha_{n1})]\cdots[\phi(t)-\phi(\alpha_{nn})]}\right|,\ t\text{ on }L_R\right],$$

because the function

$$\frac{\omega_n(z)\,[\bar\phi(\alpha_{n1})\,\phi(t)-1]\cdots[\bar\phi(\alpha_{nn})\,\phi(t)-1]}{\omega_n(t)\,[\phi(t)-\phi(\alpha_{n1})]\cdots[\phi(t)-\phi(\alpha_{nn})]}$$

is analytic in t at every point t on or exterior to L_R; its modulus for z on C and for t on or exterior to L_R is not greater than $(n+2)d_2\mu_n/d_1$.

We choose now $T_1 > T$ so that $f(z)$ is analytic in the closed region R_{T_1}, and choose $R > 1$, $R < T_1/T$. We have the familiar equation

$$(61)\qquad f(z) - r_n(z) = \frac{1}{2\pi i}\int_{L_{T_1}} \frac{\omega_n(z)\,f(t)\,dt}{\omega_n(t)\,(t-z)},\qquad z\text{ on }C.$$

By (56) we have

$$\overline{\lim_{n\to\infty}}\ \mu_n^{1/n} < T_1/T,$$

$$\lim_{n\to\infty}\left|\frac{[\phi(t)-\phi(\alpha_{n1})]\cdots[\phi(t)-\phi(\alpha_{nn})]}{[\bar\phi(\alpha_{n1})\,\phi(t)-1]\cdots[\bar\phi(\alpha_{nn})\,\phi(t)-1]}\right|^{1/n} = \frac{1}{T_1},\qquad t\text{ on }L_{T_1},$$

from which (57) follows by (60).

If inequality (40) is applied to (60) and (61), where z lies on L_Z, we have a result which implies not only (58) but also the following

COROLLARY 1. *Under the hypothesis of Theorem* 15 *we have for* $Z < T$

$$(62)\qquad \overline{\lim_{n\to\infty}}\,[\max|f(z) - r_n(z)|,\ z\text{ on or within }L_Z]^{1/n} < Z/T.$$

We can formulate at once

COROLLARY 2. *If Theorem* 15 *is modified by requiring that* $f(z)$ *shall be analytic merely interior to* R_T, *inequalities* (57) *and* (62) *are to be modified by replacing the sign* $<$ *by the sign* $\leqq$, *and equation* (58) *is valid interior to* R_T, *uniformly on any closed set interior to* R_T.

The study of convergence of other sequences of form (13) can be carried through without difficulty:

THEOREM 16. *Let C, $\phi(z)$, S, and R_T have the same significance as in Theorem 15, and let (56) be valid as before. Let $f(z)$ be analytic interior to R_T. If the functions $R_n(z)$ of form (13) satisfy the condition*

$$\overline{\lim_{n\to\infty}} [\max | f(z) - R_n(z) |, z \text{ on } C]^{1/n} \leqq 1/T, \tag{63}$$

then we have also

$$\overline{\lim_{n\to\infty}} [\max | f(z) - R_n(z) |, z \text{ on } R_Z, Z < T]^{1/n} \leqq Z/T, \tag{64}$$

so the sequence $R_n(z)$ converges interior to R_T, uniformly on any closed set interior to R_T.

If the $R_n(z)$ are given so that (63) is valid but where $f(z)$ is defined merely on C, then the sequence $R_n(z)$ converges interior to $R_{T^{1/2}}$, uniformly on any closed set interior to $R_{T^{1/2}}$, and $f(z)$ is analytic interior to $R_{T^{1/2}}$.

If the points $\alpha_{nk} = \alpha_k$ are independent of n and the $R_n(z)$ are given so that (63) is valid but where $f(z)$ is defined merely on C, then the sequence $R_n(z)$ converges interior to R_T, uniformly on any closed set interior to R_T, and $f(z)$ is analytic interior to R_T; consequently (64) is valid.

The proofs are entirely analogous to proofs given previously (§9.3) and are left to the reader. The reader should likewise have no difficulty in modifying the proof of Theorem 12 (no new Lemmas are necessary; we use the fact that for n sufficiently large the α_{nk} all lie exterior to some C_A; Theorem 16 is applicable) to yield

THEOREM 17. *Let C, $\phi(z)$, S, R_T, and the α_{nk} have the same significance as in Theorem 15, and let (56) be valid as before. Let $f(z)$ be analytic interior to R_T. Let $R_n(z)$ be the function of form (13) of best approximation to $f(z)$ on C in the sense of Tchebycheff, in the sense of least p-th powers ($p > 0$) over the boundary of C (assumed rectifiable, C consisting of a finite number of Jordan arcs or regions or both), in the sense of least p-th powers over the area of C (if C is a closed region), in the sense of least p-th powers over γ: $| w | = 1$ when K is mapped onto the exterior of γ so that the points at infinity correspond to each other, or in the sense of least p-th powers over γ: $| w | = 1$ or over the region $| w | \leqq 1$ when C is mapped onto $| w | \leqq 1$ (C being a closed region), in every case with a positive continuous weight function. Then in each case we have both (63) and (64), so the sequence $R_n(z)$ converges to $f(z)$ interior to R_T, uniformly on any closed set interior to R_T.*

The restriction that K be connected can be modified in various cases; compare §5.3. Likewise the requirement that the norm function be positive and continuous can be lightened.

Theorem 17 is a direct generalization of Theorem 7. Moreover, the case that every α_{nk} is infinite is not excluded; under such circumstances we set $\psi(z) \equiv \phi(z)$ in (56), and we obtain many of the results of Chapter V.

Theorem 17 is essentially invariant under linear transformation. To be sure,

best approximation with a norm function after transformation may not be equivalent to best approximation with the same norm function before transformation if approximation is measured by an integral, but the two problems are equivalent if the norm functions are suitably modified.

Inequality (63) cannot be greatly improved in Theorem 17 even if we assume $f(z)$ analytic in the closed region $\bar{S} + C$, where S is bounded by the curve $|\psi(z)| = T$, and provided no further restrictions are placed on the points α_{nk}. This is shown by the example given at the end of §9.4.

§9.11. Operations with asymptotic conditions

Condition (56) is in a sense a direct generalization of (17), but is far less transparent as to content. We shall therefore devote some space to the study of (56) and similar conditions. For convenience in reference we state first several lemmas. Each is proved from the familiar fact that if Γ_1 and Γ_2 are analytic Jordan curves which bound an annular region R, with Γ_2 interior to Γ_1, and if $U(x, y)$ is harmonic in $\bar{R}$, then for (x, y) in R we have

$$U(x, y) = \frac{1}{2\pi} \int_{\Gamma_1} \left(U(x', y') \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U(x', y')}{\partial \nu} \right) ds \tag{65}$$

$$+ \frac{1}{2\pi} \int_{\Gamma_2} \left(U(x', y') \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U(x', y')}{\partial \nu} \right) ds, \quad r^2 = (x - x')^2 + (y - y')^2,$$

where ν indicates exterior normal for R, and integration on Γ_1 and Γ_2 is in the positive sense with respect to R. Here and below the running coordinates are taken as (x', y'). The proofs of these lemmas are not difficult, and are left to the reader.

LEMMA I. *If Γ_2 is an analytic Jordan curve whose exterior is R, and if $U(x, y)$ is harmonic in $\bar{R}$, including the point at infinity, then for (x, y) in R we have*

$$U(x, y) = \frac{1}{2\pi} \int_{\Gamma_2} \left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu} \right) ds + U(\infty).$$

LEMMA II. *If Γ is an analytic Jordan curve, if $U(x, y) = \log [(x - x_0)^2 + (y - y_0)^2]^{1/2}$ and if both (x_0, y_0) and (x, y) lie interior to Γ, then we have*

$$\frac{1}{2\pi} \int_{\Gamma} \left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu} \right) ds = 0.$$

LEMMA III. *If Γ is an analytic Jordan curve, if $U(x, y) = \log [(x - x_0)^2 + (y - y_0)^2]^{1/2}$ and if both (x_0, y_0) and (x, y) lie exterior to Γ, then we have*

$$\frac{1}{2\pi} \int_{\Gamma} \left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu} \right) ds = 0. \tag{66}$$

More generally, equation (66) *holds whenever (x, y) lies exterior to Γ and $U(x, y)$ is harmonic on and within Γ.*

LEMMA IV. *If Γ_2 is an analytic Jordan curve whose exterior is R, if $U(x, y)$ is harmonic in $\bar{R}$ except at infinity, and if in the neighborhood of the point at infinity we have $U(x, y) = k \log [(x - x_0)^2 + (y - y_0)^2]^{1/2} + U'(x, y)$, where k is constant, U' is harmonic at infinity, and (x_0, y_0) lies interior to Γ_2, then for (x, y) in R we have*

$$U(x, y) = \frac{1}{2\pi} \int_{\Gamma_2} \left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu} \right) ds + U'(\infty) .$$

We are now in a position to state some results.

THEOREM 18. *Let the relation*

$$\lim_{n \to \infty} \left| \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \right|^{1/n} = \exp U(x, y) \tag{67}$$

be valid uniformly on an arbitrary closed set interior to some annular region R bounded by Jordan curves C_1 and C_2, with C_2 interior to C_1. Let each point β_{nk} lie on or interior to C_2 and each point α_{nk} lie on or exterior to C_1. Then the function $U(x, y)$ cannot be identically constant. At every finite point (x, y) exterior to C_2 we have

$$\begin{aligned} &\lim_{n \to \infty} | (z - \beta_{n1}) \cdots (z - \beta_{n,n+1}) |^{1/n} \\ &\quad = \exp \left[\frac{1}{2\pi} \int_{\Gamma_2} \left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu} \right) ds \right], \qquad z = x + iy , \end{aligned} \tag{68}$$

uniformly on any closed finite set exterior to C_2, where Γ_2 is an analytic Jordan curve in R containing C_2 but not (x, y) in its interior; at every point (x, y) interior to C_1 we have

$$\begin{aligned} &\lim_{n \to \infty} | (z - \alpha_{n1}) \cdots (z - \alpha_{nn}) |^{1/n} \\ &\quad = \exp \left[\frac{1}{2\pi} \int_{\Gamma_1} \left(U \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial U}{\partial \nu} \right) ds \right], \qquad z = x + iy , \end{aligned} \tag{69}$$

uniformly on any closed set interior to C_1, where Γ_1 is an analytic Jordan curve in R containing (x, y) and C_2 but not C_1 in its interior.

The functions $\log | z - \beta_{nk} |$ are uniformly limited on any closed set R' interior to R, and if the α_{nk} are uniformly limited so are the functions $\log | z - \alpha_{nk} |$ on R'. Thus the logarithm of the quantity whose limit appears in (67) is uniformly limited on R', and the sequence of such logarithms forms a normal family of harmonic functions in R. Consequently the mere existence of the limit expressed in (67) implies the uniformity of that limit. If the α_{nk} are not uniformly limited, it may be desirable to modify equations (67) and (69), for instance by inserting between bars the factor $\alpha_{n1}\alpha_{n2} \cdots \alpha_{nn}$ in (67) and its reciprocal in (69);

under these conditions, the mere existence of the limit in (67) implies the uniformity, and the modified equation (67) implies (68) and the modified (69).

Under the hypothesis of the theorem, it is clearly immaterial whether or not we include in (67) and (68) the factor $z - \beta_{n,n+1}$; the existence, uniformity, and value of the limit in (67) and (68) are the same in either case.

Let (x, y) be an arbitrary point interior to R, and let the analytic Jordan curves Γ_1 and Γ_2 in R separate (x, y) respectively from C_1 and C_2, with Γ_2 interior to Γ_1 and C_2 interior to Γ_2. If we set

$$U_n'(x, y) = \frac{1}{n}[\log|z - \beta_{n1}| + \cdots + \log|z - \beta_{nn}|],$$

$$U_n''(x, y) = \frac{1}{n}[\log|z - \alpha_{n1}| + \cdots + \log|z - \alpha_{nn}|],$$

$$U_n(x, y) = U_n'(x, y) - U_n''(x, y),$$

we have by Lemmas IV and II and by (65) and Lemma III respectively,

$$U_n'(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n'\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n'}{\partial \nu}\right)ds,$$

$$0 = \frac{1}{2\pi}\int_{\Gamma_1}\left(U_n'\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n'}{\partial \nu}\right)ds,$$

$$U_n''(x, y) = \frac{1}{2\pi}\int_{\Gamma_1}\left(U_n''\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n''}{\partial \nu}\right)ds,$$

$$0 = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n''\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n''}{\partial \nu}\right)ds;$$

the normal ν indicates exterior normal for the region bounded by Γ_1 and Γ_2. Consequently we may write

$$U_n'(x, y) = \frac{1}{2\pi}\int_{\Gamma_2}\left(U_n\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n}{\partial \nu}\right)ds,$$

(70)

$$U_n''(x, y) = \frac{1}{2\pi}\int_{\Gamma_1}\left(U_n\frac{\partial \log r}{\partial \nu} - \log r\frac{\partial U_n}{\partial \nu}\right)ds.$$

The uniformity of approach of $U_n(x', y')$ to $U(x', y')$ on any closed set interior to R implies uniformity of approach of the derivatives of $U_n(x', y')$ to the corresponding derivatives of $U(x', y')$ on any closed set interior to R, hence on Γ_1 and Γ_2, so the limits of the right-hand members in (70) exist uniformly as stated, and equations (68) and (69) follow for (x, y) in R. But the functions U_n' and U_n'' are harmonic respectively throughout the finite plane except in the points β_{nk} and α_{nk}, and are represented by equations (70) not merely when (x, y) lies in R but whenever (x, y) is a finite point exterior to Γ_2 or interior to Γ_1 respectively. The right-hand members of equations (70) approach the obvious

limits uniformly for (x, y) on any closed limited set exterior to Γ_2 or interior to Γ_1 respectively. If we note that under these restrictions on (x, y) the right-hand members of (68), (69), and (70) are independent of the particular curves Γ_1 and Γ_2 chosen, we can now conclude the validity of (68) and (69) as stated.

It remains to show that $U(x, y)$ is not identically constant. For each curve Γ_1 of the kind previously considered we have

$$(71) \qquad \lim_{n\to\infty} \int_{\Gamma_1} \frac{\partial U_n'}{\partial \nu}\, ds = \lim_{n\to\infty} \int_{\Gamma_1} \frac{\partial U_n}{\partial \nu}\, ds = \int_{\Gamma_1} \frac{\partial U}{\partial \nu}\, ds;$$

the first equality follows because U_n'' is harmonic on and interior to Γ_1, so the integral of $\partial U_n''/\partial \nu$ over Γ_1 is zero, and the second follows by the uniformity of the convergence of the derivatives of U_n on any closed set interior to R. But the expression whose limit appears in the first member of (71) is precisely 2π, so by comparing the first and last members of (71) we see that the function U cannot be identically constant, and Theorem 18 is established. The fact that U cannot be identically constant was used without proof in §9.5 in the application of Theorem 7, essentially under the present conditions.

We remark incidentally that under the conditions of Theorem 18 we have

$$(72) \qquad \begin{aligned} &\frac{1}{2\pi} \int_{\Gamma_1} \frac{\partial U_n''}{\partial \nu}\, ds = 0\,, &\qquad &\frac{1}{2\pi} \int_{\Gamma_1} \frac{\partial U''}{\partial \nu}\, ds = 0\,, \\ &\frac{1}{2\pi} \int_{\Gamma_2} \frac{\partial U_n'}{\partial \nu}\, ds = -1\,, &\qquad &\frac{1}{2\pi} \int_{\Gamma_2} \frac{\partial U'}{\partial \nu}\, ds = -1\,, \end{aligned}$$

where U' and U'' are the limits of U_n' and U_n'' respectively exterior to C_2 and interior to C_1.

One interesting consequence of Theorem 18 is that points α_{nk} or β_{nk} or both that are given implicitly can frequently be replaced by *explicit* points or more conveniently located points α_{nk} or β_{nk} or both, without altering the limits that occur in (67), (68), or (69). For instance, if Γ_2 can be chosen as a curve on which U is constant, the right-hand member of (68) can be written in the form

$$\exp\left[\frac{1}{2\pi} \int_{\Gamma_2} \log r\, d\sigma\right], \qquad\qquad d\sigma = -\frac{\partial U}{\partial \nu}\, ds\,.$$

Suppose on Γ_2 we have $d\sigma \geqq 0$ but σ not identically constant. Then new points β_{nk} can be chosen uniformly distributed on Γ_2 with respect to the parameter σ, and (68) unchanged will still be valid exterior to Γ_2, uniformly on any closed set exterior to Γ_2. Under the present circumstances we can perhaps replace the points α_{nk} by new points on Γ_1 chosen in a similar way in connection with (69), but a second term remains in the square bracket in the right-hand member of (69), namely the value of U on Γ_1, which may be different from zero. Moreover, we surely cannot use (69) with U replaced by U'', for (72) shows that $(\partial U''/\partial \nu)\, ds$ cannot be non-negative everywhere on Γ_1 unless identically zero on Γ_1. Nevertheless a more general remark applies.

Let Γ be an analytic Jordan curve, and let $V(x, y)$ be harmonic on and exterior to Γ with a logarithmic singularity at infinity in the sense of Lemma IV. Then we have for suitable k_1

$$V(x, y) = \frac{1}{2\pi} \int_\Gamma \left(V \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V}{\partial \nu} \right) ds + k_1, \quad z \text{ exterior to } \Gamma .$$

Let V_1 be a function harmonic on and interior to Γ, and either coinciding with $V(x, y)$ on Γ or differing from $V(x, y)$ on Γ by a constant. By Lemma III we have for z exterior to Γ,

$$0 = \frac{1}{2\pi} \int_\Gamma \left(V_1 \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V_1}{\partial \nu} \right) ds ,$$

$$\begin{aligned} V(x, y) &= \frac{1}{2\pi} \int_\Gamma \left[(V - V_1) \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial (V - V_1)}{\partial \nu} \right] ds + k_1 \\ &= - \frac{1}{2\pi} \int_\Gamma \log r \frac{\partial (V - V_1)}{\partial \nu} ds + k_1 . \end{aligned}$$

Here we can apply the reasoning previously used (§§7.6 and 8.7) choosing points β_{nk} uniformly distributed on Γ with respect to the parameter σ, where $d\sigma = [\partial(V_1 - V)/\partial\nu]\, ds$, provided we have $d\sigma \geqq 0$ on Γ but σ not identically constant on Γ. It may readily occur that this condition is known to be satisfied, as in the situation of §§9.4 and 9.5. For the points β_{nk} uniformly distributed on Γ with respect to σ we have

$$\lim_{n \to \infty} | (z - \beta_{n1}) \cdots (z - \beta_{nn}) |^{1/n} = e^{k_2[V(x, y) - k_1]}$$

uniformly on any closed limited point set exterior to Γ, where the constant k_2 is suitably chosen.

A similar remark applies to the obtaining of points α_{nk}. Let Γ be an analytic Jordan curve, and let $V(x, y)$ be harmonic on and within Γ:

$$V(x, y) = \frac{1}{2\pi} \int_\Gamma \left(V \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V}{\partial \nu} \right) ds , \qquad z \text{ interior to } \Gamma .$$

If $V_1(x, y)$ is harmonic on and exterior to Γ with a logarithmic singularity at infinity, then we have by (65) and Lemma IV

$$0 = \frac{1}{2\pi} \int_\Gamma \left(V_1 \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V_1}{\partial \nu} \right) ds + k_1 , \quad z \text{ interior to } \Gamma ,$$

where k_1 is suitably chosen. There results

$$V(x, y) = \frac{1}{2\pi} \int_\Gamma \left[(V - V_1) \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial (V - V_1)}{\partial \nu} \right] ds - k_1 , \; z \text{ interior to } \Gamma .$$

is identically constant on Γ, and if the points α_{nk} are
ı Γ with respect to the parameter σ, where $d\sigma$ =
we have

$$(z - \alpha_{n1}) \cdots (z - \alpha_{nn}) |^{1/n} = e^{k_2[V(x,y)+k_3]},$$

uniformly on any closed set interior to Γ, where the constants k_2 and k_3 are suitably chosen. In order for this choice of the α_{nk} to be possible, we must obviously have $d\sigma \geqq 0$ on Γ and σ not identically constant on Γ.

Whenever the function $V(x, y)$ is given, it is always possible to choose $V_1(x, y)$ so that the conditions $V_1 - V$ identically constant on Γ, $d\sigma \geqq 0$ on Γ, σ not identically constant on Γ, are satisfied; for we may choose V_2 harmonic exterior to Γ so that $V_2 - V$ is identically constant on Γ. By adding to V_2 a suitable multiple of Green's function for the exterior of Γ with pole at infinity we can ensure the conditions $d\sigma \geqq 0$, and σ not identically constant on Γ. Hence we can always choose the α_{nk} on Γ so that (73) is satisfied.

Our restriction in the present paragraph to regions bounded by only two Jordan curves and to integration over Jordan curves taken as analytic is clearly a matter of simplicity rather than compulsion; generalizations can be immediately formulated by the reader.

§9.12. Asymptotic conditions under conformal transformation

Let us return to the situation of §8.7, Theorem 9, where for simplicity we now assume that C_1 and C_2 consist of single curves, with C_2 interior to C_1. If the function $U(x, y)$ is zero on C_1, unity on C_2, harmonic interior to the region R bounded by C_1 and C_2, and continuous in $\bar{R}$, then we have used the formula

$$U(x, y) = \frac{-1}{2\pi} \int_{C_1+C_2} \log r \frac{\partial U}{\partial \nu} ds, \qquad (x, y) \text{ interior to } R,$$

where ν is the exterior normal for R. If the points α_{nk} and β_{nk} are uniformly distributed on C_1 and C_2 with respect to the parameter σ,

$$d\sigma = -(\partial U/\partial \nu)\, ds, \tag{74}$$

(of course $d\sigma$ is negative on C_2) then we have

$$\lim_{n\to\infty} \left| \frac{(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})}{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})} \right|^{1/n} = e^{qU(x,y)}, \qquad z = x + iy \text{ interior to } R, \tag{75}$$

uniformly on any closed set interior to R, where the constant q is suitably chosen. More explicitly, it is even possible to write for z in R

$$\lim_{n\to\infty} |(z - \alpha_{n1}) \cdots (z - \alpha_{nn})|^{1/n} = e^{qU_1}, \lim_{n\to\infty} |(z - \beta_{n1}) \cdots (z - \beta_{n,n+1})|^{1/n} = e^{qU_2},$$

$$U_1(x, y) = \int_{C_1} \log r\, d\sigma, \qquad U_2(x, y) = \int_{C_2} \log r\, d\sigma.$$

The function σ defined by (74) is on C_1 or C_2 precisely a function conjugate to $-U: \partial\sigma/\partial s = -\partial U/\partial \nu$, and such a function has invariant properties under conformal transformation. Under such a transformation, let R be transformed into a finite annular region whose outer boundary corresponds to C_1. Then the harmonic function $U(x, y)$ remains harmonic, and the values of the transform of $U(x, y)$ on the transforms of C_1 and C_2 are again zero and unity. Points α_{nk} and β_{nk} on C_1 and C_2 respectively which are uniformly distributed on C_1 and C_2 with respect to σ correspond under this transformation to points on the transforms of C_1 and C_2 which are uniformly distributed with respect to the transform of σ, and equation (75) holds if the original quantities z, α_{nk}, β_{nk}, $U(x, y)$ are replaced by the new ones.

THEOREM 19. *Let C_1 and C_2 be Jordan curves, with C_2 interior to C_1, and let $U(x, y)$ be harmonic in the region R bounded by C_1 and C_2. Let $U(x, y)$ be continuous in $\bar{R}$ and take on the values zero and unity on C_1 and C_2 respectively. If points α_{nk} and β_{nk} are chosen uniformly distributed on C_1 and C_2 with respect to the conjugate of $-U(x, y)$, then (75) is valid interior to R, uniformly on any closed set interior to R, where the constant q is suitably chosen.*

The invariant character of the geometric situation is clear from Theorem 19, for the statement involves in the geometric configuration only entities which are invariant under the conformal mapping of R onto an annular region R' bounded by Jordan curves C_1' and C_2', corresponding to C_1 and C_2 respectively, and with C_2' interior to C_1'. Of course the function that appears in the left-hand member of (75) is not invariant, but depends on the points α_{nk} and β_{nk} which are invariant. Even q is invariant, the quotient of -2π by the integral of $d\sigma$ over C_1, but obviously q plays a relatively minor rôle in the study of convergence of rational functions whose poles lie in the α_{nk} found by interpolation in the β_{nk}. Even if the situation of Theorem 19 is mapped conformally onto a region containing the point at infinity in its interior, equation (75) remains valid provided the constant q is suitably modified.

We have not yet established Theorem 19 in the most general case, but only in case C_1 and C_2 are *analytic* Jordan curves; the more general situation can be treated by the method of §7.6. Let Γ_1 and Γ_2 be level curves of the function $U(x, y)$, so chosen in R that they separate (x, y) from C_1 and C_2 respectively. Then we have by (65)

$$U(x, y) = \frac{1}{2\pi} \int_{\Gamma_1+\Gamma_2} \log r \, d\sigma + \mu, \tag{76}$$

where μ is the value of $U(x, y)$ on Γ_1. The function σ is continuous in $\bar{R}$, although not single-valued. Choose a unique determination of σ in $\bar{R}$, continuous except along a single Jordan arc J: σ = constant joining C_1 and C_2. When Γ_1 and Γ_2 approach monotonically C_1 and C_2 respectively, the values of $\log r$ (considered as a function of σ) on Γ_1 and Γ_2 approach uniformly, in both (x, y)

corresponding values of $\log r$ on C_1 and C_2. Then the right-... (76) approaches the right-hand member of

$$U(x, y) = \frac{1}{2\pi} \int_{C_1+C_2} \log r \, d\sigma ,$$

uniformly for (x, y) on any closed set interior to R. Hence (77) is valid, and therefore (75). To be sure, the right-hand member of (77) is not a line integral of the usual kind. But the integrand is defined and continuous in (x, y), (x', y'), and σ on C_1 and C_2, and the integral has a proper sense. The invariant nature of Theorem 19 persists even if one of the curves C_1 and C_2 degenerates to a point; in this case the function $U(x, y)$ becomes logarithmically infinite at such a point; compare the Lemmas of §9.11. In the light of the invariant property, we may say that the situation of points equally distributed studied by Fejér (§§4.3, and 7.6), which involves a limiting case of Theorem 19, is in its geometric aspect simply a conformal transformation of Runge's study (§7.1) of interpolation by polynomials in the n-th roots of unity.

The general geometric situation in Theorem 19 might be interpreted as a conformal transformation of the case where C_1 and C_2 are circles, say the circles $|z| = A > 1$ and $|z| = 1$ respectively. Typical points α_{nk} and β_{nk} uniformly distributed on C_1 and C_2 with respect to $U(x, y)$ are the n-th roots of A^n and the $(n + 1)$-st roots of unity respectively. We have as the right-hand member of (75) the function $|z/A|$. In this case, it should be noted, the development of a function analytic for $|z| \leqq R$, $1 < R < A$, converges uniformly for $|z| \leqq R$, and in fact this development converges on $|z| \leqq 1$ with the same geometric degree of convergence as the sequence of functions of best approximation on $|z| = 1$ in the sense of least squares. This fact has its analogue in much more general cases, as we shall later prove; compare also §9.5.

The following theorem is not surprising, in view of the fact (§9.11) that points α_{nk} can frequently be replaced by points uniformly distributed, and (Theorem 19) of the invariant properties of the latter points.

THEOREM 20. *Let R be an annular region bounded by Jordan curves C_1 and C_2, with C_2 interior to C_1. Suppose the points α_{nk} lie on or exterior to C_1, and that we have*

$$(78) \qquad \lim_{n\to\infty} |[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]|^{1/n} = |\Phi(z)|, \qquad z \text{ in } R,$$

uniformly on any closed set interior to R, where $w = \phi(z)$ maps the exterior of C_2 onto $|w| > 1$ so that the points at infinity correspond to each other. Then we have also

$$(79) \qquad \lim_{n\to\infty} |(z - \alpha_{n1}) \cdots (z - \alpha_{nn})|^{1/n} = |\Phi_1(z)|, \qquad z \text{ in } R,$$

uniformly on any closed set interior to R, where $\Phi_1(z)$ is suitably chosen.

Let us set

$$V_n = \frac{1}{n}\sum_{k=1}^{n} \log | \phi(z) - \phi(\alpha_{nk}) | , \qquad V'_n = \frac{1}{n}\sum_{k=1}^{n} \log | z - \alpha_{nk} | ,$$

$$V''_n = \frac{1}{n}\sum_{k=1}^{n} \log \left| \frac{\phi(z) - \phi(\alpha_{nk})}{z - \alpha_{nk}} \right| , \qquad V_n = V'_n + V''_n .$$

Let (x, y) be an arbitrary point interior to R, and let analytic Jordan curves Γ_1 and Γ_2 be chosen in R with z and C_2 interior to Γ_1, and with C_2 interior but z exterior to Γ_2. If ν denotes exterior normal we have

$$V'_n(x, y) = \frac{1}{2\pi}\int_{\Gamma_1} \left(V'_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V'_n}{\partial \nu} \right) ds$$
$$+ \frac{1}{2\pi}\int_{\Gamma_2} \left(V'_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V'_n}{\partial \nu} \right) ds , \qquad z \text{ in } R ,$$

with similar formulas for V_n and V''_n. But it follows from §9.11, Lemma III that we have

$$\int_{\Gamma_2} \left(V'_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V'_n}{\partial \nu} \right) ds = 0 ,$$

and it follows from (65) and §9.11, Lemma I that we have

$$\frac{1}{2\pi}\int_{\Gamma_1} \left(V''_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V''_n}{\partial \nu} \right) ds = - \log \Delta , \qquad \Delta = 1/| \phi'(\infty) | ,$$

for the function $\log | [\phi(z) - \phi(\alpha_{nk})]/[z - \alpha_{nk}] |$ is harmonic at all points of the extended plane exterior to C_2 and has the value $- \log \Delta$ at infinity. Hence we may write

$$V'_n = \frac{1}{2\pi}\int_{\Gamma_1} \left(V_n \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V_n}{\partial \nu} \right) ds + \log \Delta . \tag{80}$$

The function $| \phi(z) - \phi(\alpha_{nk}) |$ is uniformly bounded from zero on any closed set interior to R, so $\Phi(z)$ cannot vanish in R. The harmonic function V_n together with its derivatives converges uniformly on Γ_1, so by (80) the limit of V'_n exists uniformly on any closed set interior to Γ_1, hence uniformly on any closed set interior to C_1. Theorem 20 is established. The value of the limit of V'_n is

$$V'(x, y) = \frac{1}{2\pi}\int_{\Gamma_1} \left(V \frac{\partial \log r}{\partial \nu} - \log r \frac{\partial V}{\partial \nu} \right) ds + \log \Delta .$$

From the method of proof of Theorem 20, applied now after transformation onto the w-plane, where $w = \phi(z)$, it follows that a condition of form (79) valid in R when the points α_{nk} lie on or exterior to C_1, implies a condition of form (78), for in the proof of Theorem 20 we have not actually used the fact that $\phi(z)$ is

the mapping function. We have used only the fact that $\phi(z)$ is analytic and univalent on and exterior to C_2, and the inverse $z = \chi(w)$ has this property in the w-plane exterior to $|w| = 1$.

COROLLARY 1. *If the region R is as described in Theorem 20, and if we have (79) for z in R, uniformly on any closed set interior to R, then we also have (56) for z in R, uniformly on any closed set interior to R, where $\psi(z)$ is suitably chosen.*

We have already pointed out that (79) implies (78). But if (78) is interpreted in the w-plane, $w = \phi(z)$, it follows from the harmonic properties of the logarithms of the functions involved that we also have convergence of

$$|[\phi(\alpha_{n1}) - 1/\bar{w}] \cdots [\phi(\alpha_{nn}) - 1/\bar{w}]|^{1/n},$$

and such convergence together with (78) implies (56), convergence in every case for z interior to R, uniformly on any closed set interior to R. It follows as in the proof of Theorem 18 that $\psi(z)$ is not identically constant, but (§9.10) the function $|\psi(z)|$ approaches unity when z approaches C_2. Thus Theorem 15 applies whenever points α_{nk} are given exterior to some region containing C in its interior, such that (79) is valid in that region.

By precisely the method of proof of Theorem 20, we can formulate an analogous result:

COROLLARY 2. *Let R be an annular region bounded by Jordan curves C_1 and C_2, with C_2 interior to C_1. Suppose the points α_{nk} lie on or exterior to C_1, and that we have*

$$(81)\quad \lim_{n\to\infty}\left|\frac{[\phi(z) - \phi(\alpha_{n1})] \cdots [\phi(z) - \phi(\alpha_{nn})]}{[\bar{\phi}(\alpha_{n1})\,\phi(z) - 1] \cdots [\bar{\phi}(\alpha_{nn})\,\phi(z) - 1]}\right|^{1/n} = \exp[V(x, y)], \quad z \text{ in } R,$$

uniformly on any closed set interior to R. Then we have

$$(82)\quad \lim_{n\to\infty}\left|\frac{(z - \alpha_{n1}) \cdots (z - \alpha_{nn})}{\phi(\alpha_{n1}) \cdots \phi(\alpha_{nn})}\right|^{1/n} = \exp[V'(x, y)], \quad z \text{ in } R,$$

uniformly on any closed set interior to R, where

$$(83)\quad V'(x, y) = \frac{1}{2\pi}\int_{\Gamma_1}\left(V\frac{\partial \log r}{\partial \nu} - \log r\,\frac{\partial V}{\partial \nu}\right) ds,$$

the analytic Jordan curve Γ_1 being chosen in R so as to contain z and C_2 in its interior.

Reciprocally, previous methods show that (82) implies a condition of form (81), where $V(x, y)$ is suitably chosen; hence (82) implies (83). In particular it follows that any given set of points α_{nk} on or exterior to C_1 for which (79) or

(82) is satisfied may be replaced by an *equivalent* set α'_{nk} on some Γ_1 interior to R, equivalent in the sense that we still have the analogue of (82)

$$\lim_{n\to\infty} |(z-\alpha'_{n1})\cdots(z-\alpha'_{nn})|^{1/n} = \exp[V'(x,y)+q], \qquad z \text{ interior to } \Gamma_1,$$

uniformly for z on any closed set interior to Γ_1, where q is suitably chosen; compare §9.5. For instance, if we choose Γ_1 as a level curve of the function V, then the points α'_{nk} may be taken uniformly distributed on Γ_1 with respect to the parameter σ, $d\sigma = -(\partial V/\partial\nu)ds$, and α'_{nk} may even be chosen to be independent of n. Modification of $V'(x, y)$ in (82) by the addition of a constant term does not modify the function $V(x, y)$ in the resulting equation (81) in any way, as one may verify by the method of proof of Corollary 1.

One consequence of the remark just made is that Theorem 15 is the "best" theorem possible, in the sense that for given C, $\psi(z)$, and T' (with the locus $|\psi(z)| = T'$ in S), and for given $f(z)$ analytic interior to $R_{T'}$ but with a singularity on the locus $|\psi(z)| = T'$, there exists a sequence α_{nk} such that for no sequence $r_n(z)$ of form (13) can we have

$$\overline{\lim_{n\to\infty}}\,[\max|f(z)-r_n(z)|,\ z \text{ on } C]^{1/n} = 1/T_1 < 1/T'.$$

We may choose the points $\alpha_{nk} = \alpha_k$ independent of n on a locus $|\psi(z)| = T_2 > T'$ in R so that (56) is satisfied with the given $\psi(z)$. Then (Theorem 16) the sequence $r_n(z)$ converges interior to R_{T_1}, uniformly on any closed set interior to R_{T_1}, so $f(z)$ is analytic throughout the interior of R_{T_1}, $T_1 > T'$, contrary to hypothesis.

A further consequence of equation (83) as implied by (81) is that corresponding to given α_{nk} satisfying (81), points of interpolation β_{nk} may be chosen on C_2 (in case C_2 is a Jordan curve), uniformly distributed with respect to the parameter σ, $d\sigma = -(\partial V/\partial\nu)ds$, because on C_2 we have $V \equiv 0$, and the equation for z in R

$$(84)\qquad V(x,y) = V'(x,y) + \frac{1}{2\pi}\int_{\Gamma_2}\left(V\frac{\partial \log r}{\partial\nu} - \log r\frac{\partial V}{\partial\nu}\right)ds$$

$$= V'(x,y) + \frac{1}{2\pi}\int_{C_2}\log r\,d\sigma$$

follows from (83). Such points of interpolation lead to a sequence $r_n(z)$ found by interpolation in the points β_{nk} converging like the sequence of Theorem 15, as may be verified by an obvious modification of §8.3, Theorem 5. The modification is a consequence of a hypothesis of form (82) rather than (79). To be sure, if the point set C of Theorem 15 is not bounded by a Jordan curve, these points β_{nk} cannot be chosen on the boundary of C; but whenever $f(z)$ is given, the points β_{nk} can be chosen depending on $f(z)$, lying on some level curve near C of the function $V(x, y)$; compare §4.5. Thus Theorem 15 (and hence Theorems 16 and 17) can be proved by a method analogous to that of §4.5 (first proof of

Theorem 5), without the method of Shen. The present suggested proof has the advantage over the previous one that the points β_{nk} can be easily determined effectively, but has the disadvantage that those points depend on $f(z)$ if C is not a Jordan region.

In the light of the invariant property expressed by (83) and (84), where Γ_1 is chosen as a level curve of the function $V(x, y)$, the situation of §9.10 may be interpreted as a transformation of the situation of §9.4, where the exterior of C in §9.10 is mapped conformally onto $|w| > 1$; this interpretation is analogous to that of Fejér's theorem on points equally distributed as a conformal transformation of Runge's theorem. Of course condition (81) in its present form is invariant under arbitrary conformal mapping of the exterior of C_2.

The formulation and proof of the analogue of Theorem 20 for points of interpolation β_{nk} instead of poles α_{nk} are left to the reader. Certain results on variously located points β_{nk} may also be obtained from our results on points α_{nk} by a transformation $z' = 1/z$. Moreover, our general methods enable one to study the properties of (67) under conformal transformation; under broad conditions, equation (67) is invariant except for possible constant factors in both members.

In most of §§9.11 and 9.12 we have studied a region R bounded by two Jordan curves C_1 and C_2. The reader can hardly fail to have noticed that that choice was made only for simplicity. Much of the discussion holds too if C_2 is replaced by the boundary of any closed limited point set C not a single point whose complement is simply connected, and if R is a simply or multiply connected region whose complement has C as one of its components.

We mention some additional problems in connection with §§9.10–9.12, not considered further here because of lack of space. Suppose the situation of Theorem 15 is modified so that the complement of C is no longer simply connected. What is the analogue of (56)? What can be deduced from that analogue?

The concept of maximal convergence of a sequence $r_n(z)$ of form (13) would seem to have a meaning in case (56) is satisfied, at least in the case that the $\alpha_{nk} = \alpha_k$ are independent of n. What are its properties and implications?

§9.13. Further problems

A few further topics deserve to be mentioned in concluding this chapter.

The reader may well raise the question whether a generalization of Theorem 15 does not exist, when in the hypothesis of Theorem 15 equation (56) is replaced by knowledge of the inferior and superior limits of the left-hand member. This generalization lies immediately at hand, and includes both Theorem 10 and Theorem 15:

THEOREM 21. *Let C be a closed limited point set not a single point whose complement K is simply connected, and let $w = \phi(z)$ map K onto $|w| > 1$ so that the points at infinity correspond to each other. Let the points α_{nk} be given exterior to C, and let the relations*

$$
(85)\qquad
\begin{aligned}
&\overline{\lim_{n\to\infty}}\left|\frac{[\bar\phi(\alpha_{n1})\,\phi(z)-1]\cdots[\bar\phi(\alpha_{nn})\,\phi(z)-1]}{[\phi(z)-\phi(\alpha_{n1})]\cdots[\phi(z)-\phi(\alpha_{nn})]}\right|^{1/n}\leqq|\psi_1(z)|,\\
&\lim_{n\to\infty}\left|\frac{[\bar\phi(\alpha_{n1})\,\phi(z)-1]\cdots[\bar\phi(\alpha_{nn})\,\phi(z)-1]}{[\phi(z)-\phi(\alpha_{n1})]\cdots[\phi(z)-\phi(\alpha_{nn})]}\right|^{1/n}\geqq|\psi_2(z)|,
\end{aligned}
$$

hold uniformly on an arbitrary closed subset of some region S. Let $\overset{*}{S}+C$ be a region which contains C in its interior but contains in its interior no limit point of the α_{nk}. Let R_T' and R_T'' denote generically the respective closed regions (if existent) which contain C in their interiors, contain no point not in C or S, and are bounded by point sets on which $|\psi_1(z)| = T > 1$ and $|\psi_2(z)| = T > 1$ in S. If the function $f(z)$ is analytic in the closed region R_T'', there exists a sequence of functions $r_n(z)$ of form* (13) *such that we have*

$$(86)\qquad \overline{\lim_{n\to\infty}}\,[\max|f(z)-r_n(z)|,\ z \text{ on } C]^{1/n}\leqq 1/T,$$

$$\overline{\lim_{n\to\infty}}\,[\max|f(z)-r_n(z)|,\ z \text{ in } R_Z',\ Z<T]^{1/n}\leqq Z/T.$$

Consequently the sequence $r_n(z)$ converges to $f(z)$ uniformly on any closed set R_Z', $Z < T$.

It has seemed to the writer preferable not to make Theorem 21 fundamental, presenting Theorems 10 and 15 as corollaries, but more natural and desirable for expository purposes to proceed in the reverse order. Nevertheless, Theorem 21 does give a unified and inclusive formulation of the other results. Theorem 21 can be proved either (i) by following precisely the method used in Theorem 15 or (ii) by applying Theorem 15 itself to yield an indirect proof. For instance, method (ii) can be formulated as follows: if (86) is not true, there exists some sequence $n_1 < n_2 < n_3 < \cdots$ such that

$$(87)\qquad \lim_{k\to\infty}[\max|f(z)-r_{n_k}(z)|,\ z \text{ on } C]^{1/n_k}=1/T_1>1/T,$$

where the $r_n(z)$ are the functions defined as in the proof of Theorem 15. The functions which appear in the left-hand member of (85) form a normal family in S; from the sequence corresponding to the indices n_k can be extracted a subsequence converging uniformly on any closed set interior to S. Theorem 15 does not require the sequence $r_n(z)$ to be defined for every n, so Theorem 15 contradicts (87). An advantage of (ii) over (i) is that (ii) applies Theorem 15 itself instead of its method of proof (showing incidentally that Theorem 21 is not greatly different in content from Theorem 15); an advantage of (i) over (ii) is in not employing the principle of choice, for that principle can be avoided in (i).

* We consider such a relation as $\overline{\lim}_{n\to\infty}|F_n(z)|\leqq|F(z)|$ to hold uniformly if to every $\epsilon>0$ corresponds a number N such that $n>N$ implies uniformly $|F_n(z)|\leqq|F(z)|+\epsilon$.

The following theorem generalizes both Theorem 12 and Theorem 17, and is proved similarly to those theorems, by the theorem which is the natural generalization of both Theorem 11 and Theorem 16:

THEOREM 22. *Let the hypothesis of Theorem* 21 *be satisfied. Let $R_n(z)$ be the function of form* (13) *of best approximation to $f(z)$ on C in the sense of Tchebycheff, in the sense of least p-th powers ($p > 0$) over the boundary of C (assumed rectifiable, C consisting of a finite number of Jordan arcs or regions or both), in the sense of least p-th powers over the area of C (if C is a closed region), in the sense of least p-th powers over γ: $|w| = 1$ when K is mapped onto the exterior of γ so that the points at infinity correspond to each other, or in the sense of least p-th powers over γ: $|w| = 1$ or over the region $|w| \leqq 1$ when C is mapped onto $|w| \leqq 1$ (C being a closed region), in every case with a positive continuous norm function. Then in each case we have*

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - R_n(z)|, z \text{ on } C]^{1/n} \leqq 1/T,$$

$$\overline{\lim_{n\to\infty}}\, [\max |f(z) - R_n(z)|, z \text{ on } R'_Z, Z < T]^{1/n} \leqq Z/T,$$

so the sequence $R_n(z)$ converges to $f(z)$ uniformly on any closed set R'_Z, $Z < T$.

The general problem of best approximation by rational functions of given degree whose poles are not prescribed, or are merely prescribed to lie on a given point set, is still unsolved except in particular cases. In this problem, the poles of the approximating rational functions ordinarily depend not merely on the given point sets but also on the function approximated, and this dependence seems relatively complicated.

Let us illustrate this fact of dependence in a rather trivial case. We approximate to the function $f(z)$ on a closed set C which contains infinitely many points, by rational functions $r_n(z)$ of respective degrees n whose poles are required to lie in a closed set E; if the poles of the $r_n(z)$ are not restricted, we can choose E as the entire plane. For definiteness we may consider $f(z)$ analytic on C, best approximation measured in the sense of Tchebycheff, and suppose C is not the entire plane. If $f(z)$ is the function $1/(t - z)$, where t is a point of E exterior to C, then every function $r_n(z)$, $n \geqq 1$, clearly coincides with $f(z)$. Similarly, if $f(z)$ is a rational function of degree m whose poles lie in E exterior to C, then every function $r_n(z)$, $n \geqq m$, is identical with $f(z)$, and $r_n(z)$ has its poles precisely in the poles of $f(z)$.

This illustration indicates the dependence on $f(z)$, or at least on the singularities of $f(z)$, of the poles of the approximating functions. No methods have as yet been devised in very general cases for determining these poles in order to study in detail (with respect to degree of convergence, overconvergence, etc.) the convergence of the $r_n(z)$. It is clear, however, that the results of the present chapter are of significance in that general study; they yield at once certain conclusions regarding degree of convergence, and in case E and C have no point in common yield also conclusions regarding overconvergence.

It is appropriate to state here the following theorem [Walsh, 1931b] concerning this domain of ideas:

Let $f(z)$ be single-valued and analytic on the entire plane except in a closed reducible (§1.1) *set E. Suppose C is a closed limited point set not a single point with no point in common with E. Then there exists a sequence of rational functions $r_n(z)$ of respective degrees n whose poles lie in E such that we have*

$$\lim_{n\to\infty} [\max | f(z) - r_n(z) |, z \text{ on } C]^{1/n} = 0 .$$

Let $R_n(z)$ denote the rational function of degree n whose poles lie in E of best approximation to $f(z)$ on C in the sense of Tchebycheff (if the complement of C is regular), in the sense of least p-th powers ($p > 0$) over γ: $| w | = 1$ when the complement of C is mapped onto $| w | > 1$ (if the complement of C is simply connected), in the sense of least p-th powers over the boundary of C (C being bounded by a finite number of rectifiable Jordan arcs and curves which separate the plane into only a finite number of regions), in the sense of least p-th powers over the area of C (C having at least one interior point), or in the sense of least p-th powers over $| w | < 1$ or $| w | = 1$ after conformal mapping of C onto $| w | < 1$ (C being a simply connected region), in every case with a positive continuous norm function. Then we have

$$\lim_{n\to\infty} [\max | f(z) - R_n(z) |, z \text{ on } C]^{1/n} = 0 .$$

The set E cannot separate the plane, so (Theorem 6, Corollary 5) *the sequence $R_n(z)$ converges to the function $f(z)$ on the entire plane except on the set E. On any closed set C_1 having no point in common with E we have*

$$\lim_{n\to\infty} [\max | f(z) - R_n(z) |, z \text{ on } C_1]^{1/n} = 0 .$$

A problem related to the one just considered is that of assigning not the degree of the approximating rational function, but the degrees of the numerator and denominator, where the poles are either arbitrary or are fully prescribed, or are merely required to lie on a given point set. Here too the results and methods of the present chapter are of significance. If the poles are fully prescribed, the present methods can be modified so as to give fairly satisfactory results; compare Theorem 2 and also §9.5. If the poles are merely required to lie on a given point set, our present methods also have some application.

If the poles are entirely arbitrary, we are led to a table of double entry:

$$\begin{gathered} R_{00}(z),\ R_{01}(z),\ R_{02}(z),\ \cdots, \\ R_{10}(z),\ R_{11}(z),\ R_{12}(z),\ \cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{gathered} \tag{88}$$

where $R_{mn}(z)$ is the rational function of form

$$\frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}, \qquad b_0 z^n + b_1 z^{n-1} + \cdots + b_n \not\equiv 0, \tag{89}$$

of best approximation to $f(z)$ on C. This table is entirely analogous to the Padé table, which is constructed of functions of form (89) of highest order *interpolating* to a given $f(z)$ say at the origin rather than of best approximation. It is of interest to study the convergence of various sequences formed from the table (88); the chief difficulty is that the poles of the $R_{mn}(z)$ are not known explicitly or even asymptotically, although it is clear that those poles depend to some extent on the singularities of $f(z)$. This general problem obviously deserves further study. One of the more immediate results [Walsh, 1934] is the following theorem:

Let $f(z)$ be analytic interior to a limited Jordan region C, continuous in the corresponding closed region $\bar{C}$. Let $R_{mn}(z)$ denote the function of form (89) *of best approximation to $f(z)$ on C in the sense of Tchebycheff. Then any infinite sequence of the $R_{mn}(z)$ formed from the table* (88), *for which the inferior limit of the first subscript is not less than the number of zeros of $f(z)$ interior to C, converges to $f(z)$ uniformly in $\bar{C}$.*

CHAPTER X

INTERPOLATION AND FUNCTIONS ANALYTIC IN THE UNIT CIRCLE

§10.1. The Blaschke product

There are primarily two problems of the theory of interpolation: 1) study of the existence and uniqueness of functions of a certain class taking on prescribed values in given points; 2) representation or approximation of a given function by means of auxiliary special functions required to interpolate to it in certain points. Illustrations of 1) are §3.1, Theorem 2 and §8.1, Theorem 1; illustrations of 2) occur throughout Chapters VII and VIII. We shall undertake in the present chapter the further study of 1), particularly with reference to functions analytic interior to the unit circle C: $|z| = 1$, where the prescribed points of interpolation are also interior to C. Of central importance in this study is the *Blaschke product*, to the consideration of which we now turn.

THEOREM 1. *If the points $\beta_1, \beta_2, \cdots$ lie interior to C: $|z| = 1$, and if the product $\prod |\beta_n|$ diverges,*[*] *then the product*

$$(1) \qquad \prod_{n=1}^{\infty} \frac{\bar{\beta}_n}{|\beta_n|} \frac{z - \beta_n}{\bar{\beta}_n z - 1}$$

also diverges interior to C. On every closed set interior to C we have uniformly

$$(2) \qquad \lim_{N \to \infty} \prod_{n=1}^{N} \frac{\bar{\beta}_n}{|\beta_n|} \frac{z - \beta_n}{\bar{\beta}_n z - 1} = 0 .$$

Product (1) is called the *Blaschke product* corresponding to the numbers β_n. If $\beta_n = 0$, the corresponding factor $\bar{\beta}_n/|\beta_n|$ in (1) and (2) and elsewhere throughout the present chapter is to be replaced by unity. Indeed, so far as Theorem 1 is concerned *every* factor $\bar{\beta}_n/|\beta_n|$ can be replaced by unity, but the form that appears in (1) is more advantageous in some of our later work, for instance in Theorem 4.

Theorem 1 is proved by a method used in §9.6. The partial products of (1) are uniformly limited (of modulus less than unity) interior to C. For $z = 0$, the product (1) takes precisely the form $\prod |\beta_n|$, so equation (2) holds for the value $z = 0$. If $z = 0$ is not a β_n or a limit point of the β_n, then every limit

* An infinite product none of whose factors is zero is said to converge if and only if the product of the first N factors approaches a (finite) limit different from zero as N becomes infinite. An infinite product only a finite number of whose factors are zero is said to converge if and only if the product obtained by omitting those factors converges. An infinite product an infinite number of whose factors are zero is said to diverge.

(necessarily a uniform limit on any closed set interior to C) of an infinite sequence of partial products of (1) vanishes identically interior to C by Hurwitz's theorem (§1.2), for the partial products of (1) are different from zero in the neighborhood of $z = 0$; thus equation (2) is valid uniformly on any closed set interior to C. If $z = 0$ is a limit point of the β_n, then every limit of an infinite sequence of partial products of (1) vanishes in all the points β_n, hence being analytic interior to C vanishes identically, so (2) is again valid uniformly on any closed set interior to C. Similar reasoning applies, by regarding the derivatives at the origin of an arbitrary limit function, if an infinity of the β_n are zero.

It follows from (2), valid at every point interior to C, that (1) diverges at every point interior to C distinct from the β_n. It remains now to prove divergence in the points β_n, and to prove (2) in the case that only a finite number of the β_n are zero. Divergence of (1), divergence of $\prod | \beta_n |$, and validity of equation (2) uniformly on any closed set interior to C, are all independent of the suppression of a finite number of factors. The proof of divergence in the points β_n, and of (2) if only a finite number of the β_n are zero, goes through as before after the suppression of suitable factors if a given point β_k occurs in the sequence β_n only a finite number of times or if $\beta_k = 0$ occurs in the sequence β_n only a finite number of times. In the respective contrary cases we have by definition divergence of (1) in the points β_k and $z = 0$.

An obvious extension of the reasoning already used yields the

COROLLARY. *If equation* (2) *holds at a single point z interior to C distinct from the β_n, or if* (1) *diverges at a single point z interior to C, then* (2) *holds at every point interior to C, uniformly on any closed set interior to C. In particular $\prod | \beta_n |$ diverges.*

Whenever an infinite number of the β_n lie on a closed point set interior to C, the products $\prod | \beta_n |$ and (1) diverge, for the points β_n considered conceptually distinct have a limit point z interior to C, and equation (2) is valid at this point.

THEOREM 2. *If $f(z)$ is analytic and uniformly limited interior to C, and if $f(z)$ vanishes in each of the points β_n interior to C, with $\prod | \beta_n |$ divergent, then $f(z)$ vanishes identically interior to C.*

Let us assume $|f(z)| \leqq M$ interior to C. The function

$$\phi(z) \equiv f(z) \prod_{n=1}^{N} \frac{|\beta_n|}{\bar{\beta}_n} \frac{\bar{\beta}_n z - 1}{z - \beta_n}$$

is analytic interior to C, when suitably defined in the points β_n. When z interior to C approaches C, the modulus of the second factor on the right approaches unity and the modulus of $f(z)$ can approach no limit greater than M, so the modulus of $\phi(z)$ can approach no limit greater than M. By the Principle of Maximum we have $| \phi(z) | \leqq M$ interior to C:

$$| f(z) | \leqq M \prod_{n=1}^{N} \left| \frac{\bar{\beta}_n}{| \beta_n |} \frac{z - \beta_n}{\bar{\beta}_n z - 1} \right|, \qquad z \text{ interior to } C.$$

If we now allow N to become infinite Theorem 1 yields $f(z) \equiv 0$ interior to C.

THEOREM 3. *If the sequence of functions $f_n(z)$ is uniformly limited interior to C, and if the sequence converges in each of the points β_ν interior to C, with $\prod | \beta_\nu |$ divergent, then the sequence $f_n(z)$ converges at every point interior to C, uniformly on any closed set interior to C.*

Theorem 3 includes the second part of Theorem 1.

Any two limit functions interior to C of the sequence $f_n(z)$ must coincide in the points β_n and be uniformly limited interior to C, hence (Theorem 2) must coincide throughout the interior of C. That is to say, the sequence $f_n(z)$ converges interior to C, hence converges uniformly on any closed set interior to C.

In Theorem 3, as in Theorems 1 and 2, the points β_n or any finite or infinite number of them may all coincide, provided we make the usual convention relative to interpolation in multiple points. The convergence of the sequence $f_n(z)$ in a point β_ν of multiplicity m is thus considered to involve convergence of the sequences $f_n(\beta_\nu), f_n'(\beta_\nu), f_n''(\beta_\nu), \cdots, f_n^{(m-1)}(\beta_\nu)$.

We turn now to the study of convergent Blaschke products.

THEOREM 4. *If the points β_n lie interior to C: $| z | = 1$ and if the product $\prod | \beta_n |$ converges, then the sequence of partial products*

$$B_N(z) \equiv \prod_{n=1}^{N} \frac{\bar{\beta}_n}{| \beta_n |} \frac{z - \beta_n}{\bar{\beta}_n z - 1} \tag{3}$$

converges in the mean on the circumference C. Consequently the sequence (3) converges interior to C, uniformly on any closed set interior to C, to some function $B(z)$ analytic and of modulus not greater than unity interior to C. The limit in the mean on C of the sequence (3) is the set of (Fatou) boundary values of $B(z)$ almost everywhere on C, and is of modulus unity almost everywhere on C.

It is clear from the form of (3) that $B_N(z)$ is analytic interior to C, so the convergence of the sequence $B_N(z)$ to some function $B(z)$ analytic and of modulus not greater than unity interior to C follows from §5.8, Theorem 16 and from convergence in the mean on C of the sequence $B_N(z)$. The latter remains to be proved. We have at once

$$\int_C | B_{N+k}(z) - B_N(z) |^2 \, | dz | \tag{4}$$

$$= \int_C \left| \frac{\bar{\beta}_{N+1}\bar{\beta}_{N+2} \cdots \bar{\beta}_{N+k}}{| \beta_{N+1}\beta_{N+2} \cdots \beta_{N+k} |} \frac{(z - \beta_{N+1})(z - \beta_{N+2}) \cdots (z - \beta_{N+k})}{(\bar{\beta}_{N+1}z - 1)(\bar{\beta}_{N+2}z - 1) \cdots (\bar{\beta}_{N+k}z - 1)} - 1 \right|^2 | dz |.$$

We multiply the expression between vertical bars by its conjugate and integrate, setting $|dz| = dz/(iz)$. We notice by Cauchy's integral the relation

$$\int_C \frac{(z - \beta_{N+1})(z - \beta_{N+2}) \cdots (z - \beta_{N+k})\, dz}{(\bar\beta_{N+1}z - 1)(\bar\beta_{N+2}z - 1) \cdots (\bar\beta_{N+k}z - 1)\, iz}$$

$$= 2\pi \left[\frac{(z - \beta_{N+1}) \cdots (z - \beta_{N+k})}{(\bar\beta_{N+1}z - 1) \cdots (\bar\beta_{N+k}z - 1)}\right]_{z=0}$$

$$= 2\pi\beta_{N+1}\beta_{N+2} \cdots \beta_{N+k}.$$

The right-hand member of (4) thus reduces to

$$4\pi[1 - |\beta_{N+1}\beta_{N+2} \cdots \beta_{N+k}|].$$

The product $|\beta_{N+1}\beta_{N+2} \cdots \beta_{N+k}|$ approaches unity as N becomes infinite, by the convergence of $\prod |\beta_n|$ (the proof is immediate by taking logarithms), so the right-hand member of (4) approaches zero as N becomes infinite, and the sequence $B_n(z)$ converges in the mean on C (§6.3, Lemma) to some function $B'(z)$ which is integrable and with an integrable square on C.

The obvious inequality

$$\big|\, |B'(z)| - |B_n(z)| \,\big| \leqq |B'(z) - B_n(z)|$$

shows that the sequence $|B_n(z)|$, which has the constant value unity on C, converges in the mean to $|B'(z)|$ on C. A limit in the mean is essentially unique, so $|B'(z)|$ is unity almost everywhere on C. We still have to show that $B'(z)$ is the set of boundary values on C of the function $B(z)$ defined interior to C.

It follows from the method of §6.3, Theorem 3, Corollary 1 that we have

$$\lim_{n\to\infty} \int_C B_n(z)z^k dz = \int_C B'(z)z^k dz, \quad k = \cdots, -1, 0, 1, 2, \cdots.$$

If C_r denotes the circle $|z| = r < 1$ we have for these same values of k by the boundedness interior to C of the sequence $B_n(z)$

$$\lim_{n\to\infty} \int_C B_n(z)z^k dz = \lim_{n\to\infty} \int_{C_r} B_n(z)z^k dz = \int_{C_r} B(z)z^k dz = \int_C B(z)z^k dz,$$

where $B(z)$ on C denotes the Fatou boundary values of the limit $B(z)$ interior to C. The equations

$$\int_C B'(z)z^k dz = \int_C B(z)z^k dz, \qquad k = \cdots, -1, 0, 1, 2, \cdots,$$

are valid for the function $B'(z)$, which is known to be integrable together with its square on C, and for the bounded function $B(z)$. It follows that $B'(z)$ and $B(z)$ have the same Fourier coefficients (for $0 \leqq \theta \leqq 2\pi$, $z = e^{i\theta}$) and hence (§6.11)

that $B'(z)$ and $B(z)$ are equal almost everywhere on C. The proof of Theorem 4 is complete. Of course the convergence of the infinite product can be easily handled also by the general theory of infinite products.

Under the conditions of Theorem 4, we have actually proved the convergence of the infinite product (1) in the usual sense of that term, for the present function $B(z)$ has the modulus unity almost everywhere on C, hence cannot vanish identically interior to C. The limit of $B_N(z)$ cannot be zero except at a point β_n, and the sequence $B_N(z)$ can diverge at no point interior to C, as follows from the Corollary to Theorem 1.

Corollary. *If the product $\prod |\beta_n|$ converges and if the function $f(z)$ is analytic and of modulus not greater than M interior to C and vanishes in each point β_n, then the function $f(z)/B(z)$ is also analytic and of modulus not greater than M interior to C.*

The function $f(z)/B_n(z)$ is analytic and of modulus not greater than M interior to C by the Principle of Maximum. When n becomes infinite, this function approaches uniformly the function $f(z)/B(z)$ on any closed set interior to C; the uniformity of approach in the neighborhood of a point β_n follows from the uniformity of approach of the function $B_n(z)$ to the function $B(z)$ on any circumference interior to C which passes through no point β_k. The Corollary follows. The converse of the Corollary is obvious, that if $F(z)$ is analytic and of modulus not greater than M interior to C, and if $\prod |\beta_n|$ converges, then $F(z)B(z)$ is analytic and of modulus not greater than M interior to C and vanishes in each point β_n.

If the interior of C is transformed into itself by a transformation of the form $w = (z - \beta)/(\bar{\beta}z - 1)$, $|\beta| < 1$, it is clear from Theorems 2 and 4 that a convergent Blaschke product is transformed into a convergent Blaschke product, and that a divergent Blaschke product is transformed into a divergent Blaschke product; for the convergence of the Blaschke product is a necessary and sufficient condition that there exist a function analytic and uniformly limited interior to C which vanishes in the points β_n yet does not vanish identically; this latter property is unchanged by the transformation. In particular, a suitable transformation of this form enables us to eliminate the case (for some purposes exceptional) that points β_n lie at the origin.

A necessary and sufficient condition for the convergence of the product $\prod |\beta_n|$ is the convergence of the series

$$\sum (1 - |\beta_n|), \tag{5}$$

as is well known and easily proved by the method used in §9.6, Theorem 9.

Theorems 1, 2, and 3 are due to Blaschke [1915]; in Theorem 4 convergence of $B_N(z)$ to $B(z)$ interior to C is due to Blaschke, the fact that $B(z)$ is of modulus unity on C is due to F. Riesz [1923], and convergence in the mean on C of the sequence $B_N(z)$ is due to Walsh [1932d].

§10.2. Functions of modulus not greater than M

We shall now consider the problem of the existence and uniqueness of functions analytic and of modulus not greater than M interior to C: $|z| = 1$, which take on prescribed values in given points interior to C. The present treatment follows that of R. Nevanlinna [1919, 1929], but the methods and results are due also to Carathéodory, Fejér, Gronwall, I. Schur, Pick, and others.

Let the given points be $\beta_1, \beta_2, \cdots, \beta_n$, for the present finite in number and all distinct, and let the given functional values be $w_1^{(0)}, w_2^{(0)}, \cdots, w_n^{(0)}$. Let us denote by $w(z)$ the function whose existence and uniqueness are being studied, and let us denote by H^M the class of functions analytic and of modulus not greater than M interior to C.

The case $M = 0$ is trivial, for the only function of the class H^M vanishes identically; there exists a function $f(z)$ of class H^M taking on the values $w_k^{(0)}$ in the points β_k if and only if every $w_k^{(0)}$ is zero; the function $f(z)$ if existent is unique. Henceforth we exclude the case that M vanishes.

The case $n = 1$ can be solved at once. If we have $|w_1^{(0)}| = M$, the function $w(z) \equiv w_1^{(0)}$ satisfies the required conditions and (by the Principle of Maximum) is unique. If we have $|w_1^{(0)}| < M$, the function $w(z) \equiv w_1^{(0)}$ satisfies the required conditions and so also do an infinity of other functions; for instance it is sufficient if $w(z)$ is analytic interior to C, if $w(\beta_1) = w_1^{(0)}$, and if

$$|w(z) - w_1^{(0)}| \leqq M - |w_1^{(0)}|$$

interior to C.

Indeed, in this case $n = 1$, $|w_1^{(0)}| < M$, we set formally

$$w_1(z) \equiv M^2 \frac{w(z) - w_1^{(0)}}{M^2 - \bar{w}_1^{(0)} w(z)} \div \frac{z - \beta_1}{1 - \bar{\beta}_1 z}. \tag{6}$$

If $|w_1^{(0)}| < M$, if $w(z)$ is of class H^M, and if $w(\beta_1) = w_1^{(0)}$, then $w_1(z)$ as defined by (6) is also of class H^M. If $|w_1^{(0)}| < M$ and if $w_1(z)$ is of class H^M, then $w(z)$ as defined by (6) is also of class H^M and we have $w(\beta_1) = w_1^{(0)}$. This follows from the general properties of transformations of the form $\zeta = (z - \beta)/(1 - \bar{\beta}z)$, $|\beta| < 1$; whenever z interior to C approaches C, then ζ is also interior to C and approaches C. The transformation $w_1 = M^2(w - \gamma)/(M^2 - \bar{\gamma}w)$, $|\gamma| < M$, correspondingly transforms $|w| \leqq M$ into $|w_1| \leqq M$ and $|w| \geqq M$ into $|w_1| \geqq M$. If $|w_1^{(0)}| < M$, all solutions $w(z)$ of our given problem are therefore represented by (6), where $w_1(z)$ is an *arbitrary* function of class H^M; every function $w(z)$ represented by (6), where $w_1(z)$ is of class H^M, is a solution of the given problem.

To study the case $n > 1$ we again make use of (6) formally; the preceding remarks relative to (6) as a formal transformation are valid. If $|w_1^{(0)}| = M$, the only possible function of class H^M satisfying the first of the given conditions is $w(z) \equiv w_1^{(0)}$, and this function satisfies the other given conditions if and only if we have $w_1^{(0)} = w_2^{(0)} = \cdots = w_n^{(0)}$. If $|w_1^{(0)}| < M$, the question of

the existence and uniqueness of the desired function $w(z)$ is by (6) reduced to the question of the existence and uniqueness of a function $w_1(z)$ of class H^M which takes on in the distinct points

(7) $$\beta_k, \qquad k = 2, 3, \cdots, n,$$

the respective values

(8) $$w_k^{(1)} = M^2 \frac{w_k^{(0)} - w_1^{(0)}}{M^2 - \bar{w}_1^{(0)} w_k^{(0)}} \div \frac{\beta_k - \beta_1}{1 - \bar{\beta}_1 \beta_k}.$$

If $w_1(z)$ is an arbitrary such function, then (6) defines all possible solutions $w(z)$ of our given problem, and only such solutions.

THEOREM 5. *A necessary and sufficient condition that there exist a function $w(z)$ of class H^M which takes on the values $w_1^{(0)}, w_2^{(0)}, \cdots, w_n^{(0)}$ in the distinct points $\beta_1, \beta_2, \cdots, \beta_n$ interior to C is that we should have one of the two cases:*

(i) $$| w_1^{(0)} | = M, \qquad w_1^{(0)} = w_2^{(0)} = \cdots = w_n^{(0)};$$

or

(ii) $$| w_1^{(0)} | < M,$$

and there exists a function $w_1(z)$ of class H^M which takes on the given values (8) *in the $n - 1$ points* (7).

We have now a recurrent method for the general solution of our problem, because the problem of interpolation in n points has been reduced to the problem of interpolation in $n - 1$ points, and the problem of interpolation in a single point has already been solved.

Let us introduce the notation which is the natural extension of (6) and (8):

(9) $$w_\nu(z) \equiv M^2 \frac{w_{\nu-1}(z) - w_\nu^{(\nu-1)}}{M^2 - \bar{w}_\nu^{(\nu-1)} w_{\nu-1}(z)} \div \frac{z - \beta_\nu}{1 - \bar{\beta}_\nu z}, \qquad w_0(z) \equiv w(z),$$

(10) $$w_k^{(\nu)} = w_\nu(\beta_k) = M^2 \frac{w_k^{(\nu-1)} - w_\nu^{(\nu-1)}}{M^2 - \bar{w}_\nu^{(\nu-1)} w_k^{(\nu-1)}} \div \frac{\beta_k - \beta_\nu}{1 - \bar{\beta}_\nu \beta_k},$$

$$k = \nu + 1, \nu + 2, \cdots, n.$$

The precise solution of the general problem can now be expressed in the following form:

THEOREM 6. *A necessary and sufficient condition that there exist a function $w(z)$ of class H^M which takes on the values $w_1^{(0)}, w_2^{(0)}, \cdots, w_n^{(0)}$ in the distinct points $\beta_1, \beta_2, \cdots, \beta_n$ interior to C is that we have one of the two cases*

(i) $$| w_1^{(0)} | < M, \quad | w_2^{(1)} | < M, \cdots, \quad | w_\mu^{(\mu-1)} | < M, \quad | w_{\mu+1}^{(\mu)} | = M,$$
$$w_{\mu+1}^{(\mu)} = w_{\mu+2}^{(\mu)} = \cdots = w_n^{(\mu)};$$

or

(ii) $$|w_1^{(0)}| < M, \quad |w_2^{(1)}| < M, \cdots, \quad |w_n^{(n-1)}| < M.$$

In case (i), *the function $w(z)$ with the required properties is unique and given by the recursion formulas* (9) *and* (10), *where $w_\mu(z) \equiv w_{\mu+1}^{(\mu)}$.*

In case (ii), *the function $w(z)$ is not unique. All functions with the required properties and only such functions are given by the recursion formulas* (9) *and* (10) *where $w_n(z)$ is an arbitrary function of class H^M.*

In case (i), it follows from (9) and (10) that each of the functions $w_\mu(z)$, $w_{\mu-1}(z), \cdots, w_1(z), w(z)$ is a rational function of z which has the constant modulus M on C. The function $w(z)$ is a rational function of degree precisely μ, and of no smaller degree because the solution of (9) for $w_{\nu-1}(z)$ in terms of $w_\nu(z)$

(11) $$w_{\nu-1}(z) = M^2\left[\frac{z-\beta_\nu}{1-\bar\beta_\nu z}w_\nu(z) + w_\nu^{(\nu-1)}\right] \div \left[M^2 + \bar w_\nu^{(\nu-1)}\frac{z-\beta_\nu}{1-\bar\beta_\nu z}w_\nu(z)\right]$$

is of the form $M^2(\zeta+\gamma)/(M^2+\bar\gamma\zeta)$, $|\gamma| < M$, with $|\zeta| < M$ for z interior to C, $|\zeta| = M$ for z on C, $|\zeta| > M$ for z exterior to C, so numerator and denominator cannot vanish simultaneously.

Theorems 5 and 6 extend readily to the case that the given points β_k are not all distinct, provided we introduce the usual convention relative to interpolation in points counted multiply, and provided the formulas previously given are suitably modified.

Let us assume that the function $w(z)$ is subjected to the conditions $w(\beta_1) = w_1^{(0)}$, $w'(\beta_1) = w_2^{(0)}$. If $w(\beta_1) = w_1^{(0)}$, the condition $w'(\beta_1) = w_2^{(0)}$ is satisfied for the function $w(z)$ defined by (6) when and only when we have

(12) $$w_1(\beta_1) = M^2 w_2^{(0)}(1-|\beta_1|^2)/(M^2 - |w_1^{(0)}|^2) = w_2^{(1)},$$

provided this quotient has a meaning. A necessary and sufficient condition that the required function $w(z)$ should exist is that we have

(i) $$|w_1^{(0)}| = M, \qquad w_2^{(1)} = 0,$$

or

(ii) $$|w_1^{(0)}| < M, \qquad |w_2^{(1)}| = M,$$

or

(iii) $$|w_1^{(0)}| < M, \qquad |w_2^{(1)}| < M.$$

In case (i) the only solution is $w(z) \equiv w_1^{(0)}$. In case (ii) the only solution $w(z)$ is given by (6), where $w_1(z) \equiv w_2^{(1)}$. In case (iii) a necessary and sufficient condition that $w(z)$ should be of class H^M and should satisfy the given conditions is (6) together with $w_1(\beta_1) = w_2^{(1)}$, where $w_1(z)$ is of class H^M, or in other words that $w(z)$ should be given by (6) and

$$w_2(z) = M^2 \frac{w_1(z) - w_2^{(1)}}{M^2 - \bar{w}_2^{(1)} w_1(z)} \div \frac{z - \beta_1}{1 - \bar{\beta}_1 z},$$

where $w_2(z)$ is an arbitrary function of class H^M. That is to say, the formulas (6), (9), (10) previously used still give a complete solution of the original problem, provided that we replace (10) by (12) in case $\beta_2 = \beta_1$.

If again we have $\beta_3 = \beta_2 = \beta_1$, the quantity $w''(\beta_1) = w_3^{(0)}$ is prescribed. In the case $| w_1^{(0)} | < M$, $| w_2^{(1)} | < M$, $| w_3^{(2)} | < M$, we write as usual

$$w_3(z) = M^2 \frac{w_2(z) - w_3^{(2)}}{M^2 - \bar{w}_3^{(2)} w_2(z)} \div \frac{z - \beta_1}{1 - \bar{\beta}_1 z}$$

and proceed in a similar way. Further details concerning multiple points β_k are left to the reader; no new methods are involved.

If the given points $\beta_1, \beta_2, \cdots$ are infinite in number, Theorem 6 admits a direct extension:

THEOREM 7. *A necessary and sufficient condition that there exist a function $w(z)$ of class H^M which takes on the values $w_1^{(0)}, w_2^{(0)}, \cdots$ in the points $\beta_1, \beta_2, \cdots$ interior to C is that we have one of the two cases*

$$\text{(i)} \quad | w_1^{(0)} | < M, \quad | w_2^{(1)} | < M, \cdots, \quad | w_\mu^{(\mu-1)} | < M, \quad | w_{\mu+1}^{(\mu)} | = M,$$
$$w_{\mu+1}^{(\mu)} = w_{\mu+2}^{(\mu)} = w_{\mu+3}^{(\mu)} = \cdots;$$

or

$$\text{(ii)} \quad | w_1^{(0)} | < M, \quad | w_2^{(1)} | < M, \cdots.$$

In case (i), *the function $w(z)$ with the required properties is unique and given by the recursion formulas* (9) *and* (10), *where $w_\mu(z) \equiv w_{\mu+1}^{(\mu)}$.*

In case (ii), the question of the uniqueness of the function $w(z)$ is more delicate, and will be discussed in §10.6.

The conditions of Theorem 7 are necessary, for if the present function $w(z)$ exists, then the function $w(z)$ of Theorem 6 also exists for every value of n, whether the β_n are distinct or not, so the condition of Theorem 6 is satisfied for every n.

The conditions of Theorem 7 are sufficient, for if they are satisfied the conditions of Theorem 6 are satisfied for every value of n; let us denote by $W_n(z)$ a function of class H^M which takes on the values $w_1^{(0)}, w_2^{(0)}, \quad , w_n^{(0)}$ in the points $\beta_1, \beta_2, \cdots, \beta_n$. The functions $W_n(z)$ are uniformly limited and form a normal family interior to C; any limit function of the family satisfies the conditions of interpolation in *all* the points $\beta_1, \beta_2, \cdots$, distinct or not, by virtue of the uniformity of the convergence interior to C, and is of class H^M. Theorem 7 is completely proved.

§10.3. Functions of least maximum modulus

In §10.2 we studied the existence of functions satisfying conditions of interpolation and belonging to class H^M, where M is given. We now wish to study [Kakeya, 1915; Walsh, 1930] the dependence of such functions on M, especially the determination of the smallest M for which there exists a function of class H^M satisfying given conditions of interpolation.

Let us state for reference the following lemma, a consequence of the Lemma of §9.2 by the use of an obvious substitution:

LEMMA. *The inequalities* $|\zeta| < 1$, $|\beta| < 1$, *imply*

$$\left|\frac{\zeta - \beta}{1 - \bar{\beta}\zeta}\right| \leqq \frac{|\zeta| + |\beta|}{1 + |\beta\zeta|} < 1;$$

the inequalities $|\zeta| < M$, $|\beta| < M$ *imply*

$$\left|\frac{\zeta - \beta}{M^2 - \bar{\beta}\zeta}\right| \leqq \frac{|\zeta| + |\beta|}{M^2 + |\beta\zeta|} < \frac{1}{M}.$$

We shall now prove

THEOREM 8. *Let the values* $w_1^{(0)}, w_2^{(0)}, \cdots, w_n^{(0)}$ *and the distinct points* $\beta_1, \beta_2, \cdots, \beta_n$ *be given interior to* C. *Then there is a smallest* M *such that a function* $w(z)$ *of class* H^M *exists which takes on the values* $w_k^{(0)}$ *in the points* β_k. *A necessary and sufficient condition for this smallest* M *is that case* (i) *of Theorem* 6 *should occur, and for this smallest* M *the function* $w(z)$ *is unique.*

If all of the numbers $w_k^{(0)}$ are zero, the smallest M exists and is equal to zero; the conclusion of Theorem 8 is satisfied. This trivial case is henceforth excluded.

There exists a polynomial which satisfies the given conditions of interpolation, so there exists some number M such that a function $w(z)$ satisfying the equations $w(\beta_k) = w_k^{(0)}$ belongs to class H^M. Let $M_1 > M_2 > \cdots$ be a sequence of such numbers approaching the greatest lower bound M_0 of all such numbers, and let the functions $W_1(z), W_2(z), \cdots$ satisfy the given conditions and belong to the respective classes $H^{M_1}, H^{M_2}, \cdots$. The functions $W_n(z)$ form a normal family interior to C; any limit function of the family satisfies the given conditions and is of class H^{M_0}. Thus we have established the existence of a smallest M.

For the particular $M = M_0$ whose existence we have just proved, we must have either (i) or (ii) of Theorem 6, for the function $w(z)$ exists. We show that case (ii) leads to a contradiction and is therefore impossible. Let us set the arbitrary function $w_n(z)$ identically equal to zero, or what is the same thing, set $w_{n-1}(z)$ identically equal to $w_n^{(n-1)}$, whose modulus is less than M. The Lemma with equation (11) taken successively for $\nu = n - 1, n - 2, \cdots, 1$, then yields

$$|w_{n-2}(z)| \leqq M^2 \frac{|w_n^{(n-1)}| + |w_{n-1}^{(n-2)}|}{M^2 + |w_n^{(n-1)} w_{n-1}^{(n-2)}|} = M' < M, \qquad |z| < 1,$$

$$|w_{n-3}(z)| \leqq M^2 \frac{M' + |w_{n-2}^{(n-3)}|}{M^2 + M'|w_{n-2}^{(n-3)}|} = M'' < M, \qquad |z| < 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$|w(z)| \leqq M^{(n-1)} < M, \qquad |z| < 1,$$

contrary to the properties of M as a greatest lower bound.

It follows, then, that case (i) must appear, and hence that the function $w(z)$ whose modulus does not exceed M is unique. Conversely, if case (i) occurs, the number M must be the smallest for which a function $w(z)$ can exist; if a function $\bar{w}(z)$ existed satisfying the given conditions and the inequality $|\bar{w}(z)| \leqq \bar{M} < M$ interior to C, there would exist an infinity of functions of the form

$$w(z) \equiv \bar{w}(z) + \epsilon(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)$$

satisfying the given conditions and such that $|w(z)| \leqq M$ interior to C, in contradiction with Theorem 6. The proof is complete.

Theorem 8 yields an algebraic equation for the determination of the smallest M, namely the equation $|w_n^{(n-1)}| = M$. This equation, when suitably expressed by means of (10) and (9), involves besides M merely the given quantities $w_k^{(0)}$ and β_k. To be sure, the equation is not immediately an algebraic equation in M, but it becomes such when the absolute value signs are eliminated. The number M of Theorem 8 is not necessarily the smallest positive root of this algebraic equation, for the equation does not automatically imply the inequalities that appear in (i), Theorem 6.* The number M of Theorem 8 is (by Theorem 8) the only root of this algebraic equation for which the inequalities of (i), Theorem 6, are fulfilled.

In setting up this algebraic equation, it is essential to note that if case (i) of Theorem 6 occurs with $\mu < n - 1$, the numbers $w_{\mu+2}^{(\mu+1)}, w_{\mu+3}^{(\mu+2)}, \cdots, w_n^{(n-1)}$ are not defined by (10), for the right-hand member of (10) takes the meaningless form $0/0$. Nevertheless, the equation $|w_n^{(n-1)}| = M$, where M is unknown, does have a meaning, and that equation is satisfied by the least M, if the equation is written in the form of a polynomial in M set equal to zero

Our restriction in Theorem 8 that the points β_k be unique is clearly unessential, merely a question of simplicity of proof.

We consider next the analogue [Walsh, 1935a] of Theorem 8 for an infinity of conditions of interpolation:

THEOREM 9. *Let the values $w_1^{(0)}, w_2^{(0)}, \cdots$, and the points $\beta_1, \beta_2, \cdots$ interior to C be given. Let M_n denote the smallest number such that a function of class H^{M_n} exists which takes on the values $w_1^{(0)}, w_2^{(0)}, \cdots, w_n^{(0)}$ in the points $\beta_1, \beta_2, \cdots, \beta_n$. A necessary and sufficient condition that there exist a function*

* This is shown by simple examples, such as $w_1 = 2$, $w_2 = 5/4$, $\beta_1 = 0$, $\beta_2 = 1/2$.